Stability, Dissipativity, and Optimal Control of Discontinuous Dynamical Systems

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To my parents Fuad Sadikhov and Naila Sadikhova,
and to the memory of my uncle Professor Mahammad Akhundov

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Summary

Discontinuous dynamical systems and multiagent systems are encountered in numerous engineering applications. This dissertation develops stability and dissipativity of nonlinear dynamical systems with discontinuous right-hand sides, optimality of discontinuous feedback controllers for Filippov dynamical systems, almost consensus protocols for multiagent systems with innaccurate sensor measurements, and adaptive estimation algorithms using multiagent network identifiers.

In particular, we present stability results for discontinuous dynamical systems using nonsmooth Lyapunov theory. Then, we develop a constructive feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives. Furthermore, we develop dissipativity notions and extended Kalman-Yakubovich-Popov conditions and apply these results to develop feedback interconnection stability results for discontinuous systems. In addition, we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal discontinuous feedback regulators that minimize a nonlinear-nonquadratic performance functional for Filippov dynamical systems. Then, we provide connections between dissipativity and optimality of nonlinear discontinuous controllers for Filippov dynamical systems.

Furthermore, we address the consensus problem for a group of agent robots with uncertain interagent measurement data, and show that the agents reach an almost

consensus state and converge to a set centered at the centroid of agents' initial locations. Finally, we develop an adaptive estimation framework predicated on multiagent network identifiers with undirected and directed graph topologies that identifies the system state and plant parameters online.

The consideration of nonsmooth Lyapunov functions for proving stability of feedback discontinuous systems is an important extension to classical stability theory since there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein to show the existence of a feedback stabilizing controller. A constructive feedback control law based on a universal construction of smooth control Lyapunov functions was given by Sontag. Even though a stabilizing continuous feedback controller guarantees the existence of a smooth control Lyapunov function, many systems that possess smooth control Lyapunov functions do not necessarily admit a continuous stabilizing feedback controller. However, the existence of a control Lyapunov function allows for the design of a stabilizing feedback controller that admits Filippov and Krasovskii closed-loop system solutions. In this dissertation, we develop a constructive feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives.

Furthermore, we develop dissipativity notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves. Moreover, extended Kalman-Yakubovich-Popov conditions, in terms of the discontinuous system dynamics, characterizing dissipativity via generalized Clarke gradients and locally Lipschitz continuous storage functions are derived. Finally, these results are then used to develop feedback interconnection stability results for discontinuous systems thereby providing a generalization of the small gain and positivity theorems to systems with discontinuous vector fields.

Then, we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal discontinuous feedback regulators that minimize a nonlinear-nonquadratic performance functional for Filippov dynamical systems. Furthermore, using the newly developed dissipativity notions we develop a return difference inequality to provide connections between dissipativity and optimality of nonlinear discontinuous controllers for Filippov dynamical systems. Specifically, using the extended Kalman–Yakubovich–Popov conditions we show that our discontinuous feedback control law satisfies a return difference inequality if and only if the controller is dissipative with respect to a quadratic supply rate.

One of the main challenges in robotics applications is dealing with inaccurate sensor data. Specifically, for a group of mobile robots the measurement of the exact location of the other robots relative to a particular robot is often inaccurate due to sensor measurement uncertainty or detrimental environmental conditions. In this dissertation, we address the consensus problem for a group of agent robots with uncertain interagent measurement data. Using agent location uncertainty characterized by norm bounds centered at the neighboring agent's exact locations, we show that the agents reach an almost consensus state and converge to a set centered at the centroid of agents' initial locations. The diameter of the set is shown to be dependant on the graph Laplacian and the magnitude of the uncertainty norm bound. Furthermore, we show that if the network is all-to-all connected and the measurement uncertainty is characterized by ball of radius r, then the diameter of the set to which the agents converge is $2r$. Finally, we also formulate our problem using set-valued analysis and use a set-valued invariance principle to obtain set-valued consensus protocols.

Finally, we present an adaptive estimation framework predicated on multiagent network identifiers with undirected and directed graph topologies. Specifically, the system state and plant parameters are identified online using N agents implementing adaptive observers with an interagent communication architecture. The adaptive observer architecture includes an additive term which involves a penalty on the mismatch between the state and parameter estimates. The proposed architecture is shown to guarantee state and parameter estimate consensus. Furthermore, the proposed adaptive identifier architecture provides a measure of agreement of the state and parameter estimates that is independent of the network topology and guarantees that the deviation from the mean estimate for both the state and parameter estimates converge to zero.

Chapter 1

Introduction

Numerous engineering applications give rise to discontinuous dynamical systems. Specifically, in impact mechanics the motion of a dynamical system is subject to velocity jumps and force discontinuities leading to nonsmooth dynamical systems [9,58]. In mechanical systems subject to unilateral constraints on system positions [57], discontinuities occur naturally through system-environment interactions. Alternatively, control of networks and control over networks with dynamic topologies also give rise to discontinuous systems [42]. Specifically, link failures or creations in network systems result in switchings of the communication topology leading to dynamical systems with discontinuous right-hand sides. In addition, open-loop and feedback controllers also give rise to discontinuous dynamical systems. In particular, bang-bang controllers discontinuously switch between maximum and minimum control input values to generate minimum-time system trajectories [1], whereas sliding mode controllers [22, 73] use discontinuous feedback control for system stabilization. In switched systems [8, 39], switching algorithms are used to select an appropriate plant (or controller) from a given finite parameterized family of plants (or controllers) giving rise to discontinuous systems.

In the case where the vector field defining the dynamical system is a discontinuous function of the state, system stability can be analyzed using nonsmooth Lyapunov theory involving concepts such as weak and strong stability notions, differential inclusions, and generalized gradients of locally Lipschitz continuous functions and proximal subdifferentials of lower semicontinuous functions [15]. The consideration of nonsmooth Lyapunov functions for proving stability of discontinuous systems is an important extension to classical stability theory since, as shown in [67], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory.

Many physical and engineering systems are *open systems*, that is, the system behaviour is described by an evolution law that involves the system state and the system input with, possibly, an output equation wherein past trajectories together with the knowledge of any inputs define future trajectories (uniquely or nonuniquely) and the system output depends on the instantaneous (present) values of the system state. Dissipativity theory is a system-theoretic concept that provides a powerful framework for the analysis and control design of open dynamical systems based on generalized system energy considerations. In particular, dissipativity theory exploits the notion that numerous physical dynamical systems have certain input-output and state properties related to conservation, dissipation, and transport of mass and energy. Such conservation laws are prevalent in dynamical systems, in general, and feedback control systems, in particular. The dissipation hypothesis on dynamical systems results in a fundamental constraint on the system dynamical behavior, wherein the stored energy of a dissipative dynamical system is at most equal to sum of the initial energy stored in the system and the total externally supplied energy to the system. Thus, the energy that can be extracted from the system through its input-output ports is less than or equal to the initial energy stored in the system, and hence, there can be no internal creation of energy; only conservation or dissipation of energy is possible.

The key foundation in developing dissipativity theory for nonlinear dynamical systems with continuously differentiable flows was presented by Willems [74,75] in his seminal two-part paper on dissipative dynamical systems. In particular, Willems [74] introduced the definition of dissipativity for general nonlinear dynamical systems in terms of a *dissipation inequality* involving a generalized system power input, or *supply rate*, and a generalized energy function, or *storage function*. The dissipation inequality implies that the increase in generalized system energy over a given time interval cannot exceed the generalized energy supply delivered to the system during this time interval. The set of all possible system storage functions is convex and every system storage function is bounded from below by the *available system storage* and bounded from above by the *required energy supply*.

In light of the fact that energy notions involving conservation, dissipation, and transport also arise naturally for discontinuous systems, it seems natural that dissipativity theory can play a key role in the analysis and control design of discontinuous dynamical systems. Specifically, as in the analysis of continuous dynamical systems with continuously differentiable flows, dissipativity theory for discontinuous dynamical systems can involve conditions on system parameters that render an input, state, and output system dissipative. In addition, robust stability for discontinuous dynamical systems can be analyzed by viewing a discontinuous dynamical system as an interconnection of discontinuous dissipative dynamical subsystems. Alternatively, discontinuous dissipativity theory can be used to design discontinuous feedback controllers that add dissipation and guarantee stability robustness allowing discontinuous stabilization to be understood in physical terms. As for dynamical systems with continuously differentiable flows [31], dissipativity theory can play a fundamental role in addressing robustness, disturbance rejection, stability of feedback interconnections, and optimality for discontinuous dynamical systems.

Even though passivity notions for the specific problem of the control of mechanical systems with discontinuous friction-type nonlinearities are considered in [20, 44, 77] using input-to-state stability notions and set-valued nonlinearity extensions of the circle and Popov criterion, the general problem of dissipativity theory in the sense of Willems [74,75] for discontinuous dynamical systems and its connections to nonlinear discontinuous feedback regulator theory and inverse optimal control have not been addressed in the literature. It is important to note, however, that the problem of stabilization for discontinuous systems with nonsmooth control Lyapunov functions has been extensively addressed in the literature; see [3, 5, 13, 45, 47, 69] and the references therein. However, with the exception of [7,63] that address the specific problem of $\mathcal{L}_2$-gain stabilizability, these results do not explore the underlying connections between steady-state viscosity supersolutions of the Hamilton-Jacobi-Bellman equation and nonsmooth closed-loop Lyapunov functions for guaranteeing both stability and optimality for discontinuous dynamical systems. In addition, gain, sector, and disk margin guarantees are not provided in the aforementioned references by exploiting connections between dissipativity theory, discontinuous nonlinear regulator theory, and an inverse optimal control problem.

In this dissertation, we present several results from the literature on Lyapunov-based tests for Lyapunov and asymptotic stability for nonlinear dynamical systems with discontinuous right-hand sides. Moreover, using an extended notion of control Lyapunov functions [3] we develop a universal feedback controller for discontinuous dynamical systems based on the existance of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives.

Next, we extend the results of [36] to develop dissipativity notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves. Moreover, we develop extended Kalman-Yakubovich-Popov conditions in terms of the discontinuous system dynamics for characterizing dissipativity via generalized Clarke gradients of locally Lipschitz continuous storage functions. In addition, using the concepts of dissipativity for discontinuous dynamical systems with appropriate storage functions and supply rates, we construct nonsmooth Lyapunov functions for discontinuous feedback systems by appropriately combining the storage functions for the forward and feedback subsystems. General stability criteria are given for Lyapunov, asymptotic and exponential stability for feedback interconnections of discontinuous dynamical systems. In the case where the supply rate involves the net system power or weighted input-output energy, these results provide extensions of the positivity and small gain theorems to discontinuous dynamical systems.

Furthermore, we consider a notion of optimality that is directly related to a given nonsmooth Lyapunov function. Specifically, an optimal control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal discontinuous feedback controller. In addition, we develop sufficient conditions for gain, sector, and disk margin guarantees for Filippov nonlinear dynamical systems controlled by optimal and inverse optimal discontinuous regulators. Furthermore, we develop a counterpart to the classical return difference inequality for continuous-time systems with continuously differentiable flows [11, 52] for Filippov dynamical systems and provide connections between dissipativity and optimality for discontinuous nonlinear controllers. In particular, we show an equivalence between dissipativity and optimality of discontinuous controllers holds for Filippov dynamical systems. Specifically, we show that an optimal nonlinear controller $\phi(x)$ satisfying a return difference condition is equivalent to the fact that the Filippov dynamical system with input

u and output $y = -\phi(x)$ is dissipative with respect to a supply rate of the form $[u+y]^{\mathrm{T}}[u+y] - u^{\mathrm{T}}u$.

Next, we consider a multiagent consensus problem in which agents possess sensors with limited accuracy. Modern military and national command and control infrastructure capabilities involve large-scale multilayered network systems placing stringent demands on controller design and implementation of increasing complexity. In numerous large-scale network system applications, agents can detect the location of the neighboring agents only approximately. This problem can arise in network defense systems involving low sensor quality, sensor failure, or detrimental environmental conditions resulting from a weapons of mass destruction (WMD) attack. This problem also arises in many robotics applications with inaccurate sensor data as well as low-cost, small-sized unmanned vehicles with relatively cheap sensors. In such a setting, it is desirable that the agents reach consensus approximately.

In this dissertation, we develop consensus control protocols for continuous- and discrete-time network systems that guarantee that the agents reach an almost consensus state and converge to a set centered at the centroid of the agents' initial locations. For discrete-time network systems, we also use difference inclusions and set-valued analysis to describe the inaccurate sensor measurement problem formulation.

Finally, we consider the problem of adaptive estimation of a linear system with unknown plant and input matrices. In particular, we propose a novel distributed observer architecture that adaptively identifies the dynamic system matrices using a group of N agents. Each agent generates its own adaptive identifier which is based on the identifier architecture presented in [53]. The adaptive estimation architecture builds on the work of [71] on adaptive consensus control of multiagent systems with the key difference being that the mismatch between the state and parameter estimates is also penalized, and thus, accounting for interagent communication constraints.

1.1. Brief Outline of the Dissertation

In this dissertation, we develop dissipativity theory and analyze optimality of discontinuous feedback controllers for nonlinear differential equations with discontinuous right-hand sides. Furthermore, we develop almost consensus protocols for multiagent systems with innaccurate sensor measurements and design a distributed adaptive estimation framework using multiagent network identifiers. The contents of the dissertation are as follows. In Chapter 2, we provide mathematical preliminaries and present a summary of the recent results on the stability of discontinuous dynamical systems as well as an overview of consensus problem over static networks.

In Chapter 3, we design a constructive feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives. In Chapter 4, we develop dissipativity notions and extended Kalman-Yakubovich-Popov conditions and apply these results to develop feedback interconnection stability results for discontinuous systems. In Chapter 5, we present guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal discontinuous feedback regulators and provide connections between dissipativity and optimality of nonlinear discontinuous controllers for Filippov dynamical systems.

In Chapter 6, we develop consensus control protocols for continuous- and discrete-time network systems in which agents possess sensors with limited accuracy that guarantee that the agents reach an almost consensus state and converge to a set centered at the centroid of the agents' initial locations. In Chapter 7, we design a novel distributed observer architecture using a group of N agents that adaptively identifies the dynamic system matrices of a linear system with unknown plant and input matrices. Finally, in Chapter 8, we discuss ongoing research and future extensions of the research.

Chapter 2

Mathematical Preliminaries, Stability of Discontinuous Dynamical Systems, and Consensus Over Networks

2.1. Notation and Mathematical Preliminaries

The notation used in this dissertation is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ column vectors, $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, and $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $\otimes$ denotes Kronecker product, $\oplus$ denotes Kronecker sum, and I_n or I denotes the $n \times n$ identity matrix. Furthermore, $\mathcal{L}_2$ denotes the space of all real square-integrable Lebesgue measurable (vector or matrix) functions on $[0, \infty)$ and $\mathcal{L}_\infty$ denotes the space of all real bounded Lebesgue measurable (vector or matrix) functions on $[0, \infty)$.

We write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M, $\sigma_{\max}(M)$ for the maximum singular value of the matrix M, $\rho(M)$ for the spectral radius of the matrix M, $\mathrm{spec}(M)$ for the spectrum of the square matrix M including multiplicity, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathrm{tr}(\cdot)$ for the trace operator, $V'(x)$ for the Fréchet derivative of V at x, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the *open ball centered* at α with *radius* ε, $\mathrm{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ (resp., $x(k) \to \mathcal{M}$ as $k \to \infty$, $k \in \overline{\mathbb{Z}}_+$) to denote that the trajectory $x(t)$ (resp., $x(k)$) approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists $T > t_0$ (resp., $N_0 > 0$) such that $\text{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$ (resp., $k > N_0$). Furthermore, we write $\partial\mathcal{S}$ and $\overline{\mathcal{S}}$ to denote the boundary and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively.

Moreover, in this dissertation, we distinguish between the set inclusions $\subset$ and $\subseteq$; namely, $\subset$ denotes a strict inclusion, whereas $\subseteq$ denotes a nonstrict inclusion. In addition, we use the Minkowski sum for summation of sets with an analogous definition for set subtraction. Namely, for the sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} - \mathcal{Y}$ denote, respectively, the set of all vectors $z \in \mathbb{R}^n$ such that $z = x + y$ and $z = x - y$, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Finally, the notions of openness, convergence, continuity, and compactness that we use throughout the dissertation refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$.

In this dissertation, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{2.1}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $f : \mathcal{D} \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [24] with respect to x, that is, f is bounded on a bounded neighborhood of every point x, excluding sets of measure zero, and admits an equilibrium point at $x_\text{e} \in \mathcal{D}$; that is, $f(x_\text{e}) = 0$.

An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^n$ is said to be a *Filippov solution* [24] of (2.1) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[f](x(t)), \quad \text{a.e.} \quad t \in [t_0, \tau], \tag{2.2}$$

where the *Filippov set-valued map* $\mathcal{K}[f] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\mathcal{K}[f](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S}) = 0} \overline{\text{co}}\{f(\mathcal{B}_\delta(x) \backslash \mathcal{S}\}, \quad x \in \mathbb{R}^n, \tag{2.3}$$

$2^{\mathbb{R}^n}$ denotes the collection of all subsets of $\mathbb{R}^n$, $\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, "$\overline{\text{co}}$" denotes convex closure, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero.[1] Note that since f is locally essentially bounded, $\mathcal{K}[f](\cdot)$ is upper semicontinuous and has nonempty, compact, and convex values. Thus, Filippov solutions are limits of solutions to $\mathcal{G}$ with f averaged over progressively smaller neighborhoods around the solution point, and hence, allow solutions to be defined at points where f itself is not defined. Hence, the tangent vector to a Filippov solution, when it exists, lies in the convex closure of the limiting values of the system vector field $f(\cdot)$ in progressively smaller neighborhoods around the solution point. Dynamical systems of the form given by (2.1) are called *differential inclusions* in the literature [4] and, for every state $x \in \mathbb{R}^n$, they specify a *set* of possible evolutions of $\mathcal{G}$ rather than a single one.

Since the Filippov set-valued map given by (2.3) is upper semicontinuous with nonempty, convex, and compact values, and $\mathcal{K}[f](\cdot)$ is also locally bounded [24, p. 85], it follows that Filippov solutions to (2.1) exist [24, Thm. 1, p. 77]. Recall that the Filippov solution $t \mapsto x(t)$ to (2.1) is a *right maximal solution* if it cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal Filippov solutions to (2.1) exist on $[t_0, \infty)$, and hence, we assume that (2.1) is *forward complete*. Recall that (2.1) is forward complete if and only if the Filippov solutions to (2.1) are uniformly globally sliding time stable [72, Lem 1, p. 182]. An *equilibrium point* of (2.1) is a point $x_e \in \mathbb{R}^n$ such that $0 \in \mathcal{K}[f](x_e)$. It is easy to see that x_e is an equilibrium point of (2.1) if and only if the constant function $x(\cdot) = x_e$ is a Filippov solution of (2.1). We denote the set of equilibrium points of (2.1) by $\mathcal{E}$. Since the set-valued map $\mathcal{K}[f](\cdot)$ is upper semicontinuous, it follows that $\mathcal{E}$ is closed.

[1] Alternatively, we can consider Krasovskii solutions of (2.1) wherein the possible misbehavior of the derivative of the state on null measure sets is not ignored; that is, $\mathcal{K}[f](x)$ is replaced with $\mathcal{K}[f](x) = \bigcap_{\delta>0} \overline{\text{co}}\{f(\mathcal{B}_\delta(x))\}$ and where f is assumed to be locally bounded.

To develop stability properties for discontinuous dynamical systems given by (2.1), we need to introduce the notion of generalized derivatives and gradients. Here we focus on Clarke generalized derivatives and gradients [13].

Definition 2.1.1. ([13], [5]) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. The *Clarke upper generalized derivative* of $V(\cdot)$ at x in the direction of $v \in \mathbb{R}^n$ is defined by

$$V^o(x, v) \triangleq \limsup_{y \to x, h \to 0^+} \frac{V(y + hv) - V(y)}{h}. \tag{2.4}$$

The *Clarke generalized gradient* $\partial V : \mathbb{R}^n \to 2^{\mathbb{R}^{1 \times n}}$ of $V(\cdot)$ at x is the set

$$\partial V(x) \triangleq \operatorname{co}\left\{ \lim_{i \to \infty} \nabla V(x_i) : x_i \to x, \, x_i \notin \mathcal{N} \cup \mathcal{S} \right\}, \tag{2.5}$$

where co denotes the convex hull, ∇ denotes the nabla operator, $\mathcal{N}$ is the set of measure zero of points where ∇V does not exist, $\mathcal{S}$ is any subset of $\mathbb{R}^n$ of measure zero, and the increasing unbounded sequence $\{x_i\}_{i \in \overline{\mathbb{Z}}_+} \subset \mathbb{R}^n$ converges to $x \in \mathbb{R}^n$.

Note that (2.4) always exists. Furthermore, note that it follows from Definition 2.1.1 that the generalized gradient of V at x consists of all convex combinations of all the possible limits of the gradient at neighboring points where V is differentiable. In addition, note that since $V(\cdot)$ is Lipschitz continuous, it follows from Rademacher's theorem [23, Thm 6, p. 281] that the gradient $\nabla V(\cdot)$ of $V(\cdot)$ exists almost everywhere, and hence, $\nabla V(\cdot)$ is bounded. Specifically, for every $x \in \mathbb{R}^n$, every $\varepsilon > 0$, and every Lipschitz constant L for V on $\overline{\mathcal{B}}_\varepsilon(x)$, $\partial V(x) \subseteq \overline{\mathcal{B}}_L(0)$. Thus, since for every $x \in \mathbb{R}^n$, $\partial V(x)$ is convex, closed, and bounded, it follows that $\partial V(x)$ is compact.

In order to state the main results of this dissertation, we need some additional notation and definitions. Given a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the *set-valued Lie derivative* $\mathcal{L}_f V : \mathbb{R}^n \to 2^{\mathbb{R}}$ of V with respect to f at x [5, 16] is defined as

$$\mathcal{L}_f V(x) \triangleq \left\{ a \in \mathbb{R} : \text{there exists } v \in \mathcal{K}[f](x) \text{ such that } p^{\mathrm{T}} v = a \text{ for all } p^{\mathrm{T}} \in \partial V(x) \right\}$$
$$\subseteq \bigcap_{p^{\mathrm{T}} \in \partial V(x)} p^{\mathrm{T}} \mathcal{K}[f](x). \tag{2.6}$$

If $\mathcal{K}[f](x)$ is convex with compact values, then $\mathcal{L}_f V(x)$, $x \in \mathbb{R}^n$, is a closed and bounded, possibly empty, interval in $\mathbb{R}$. If $V(\cdot)$ is continuously differentiable at x, then $\mathcal{L}_f V(x) = \{\nabla V(x) \cdot v : v \in \mathcal{K}[f](x)\}$. In the case where $\mathcal{L}_f V(x)$ is nonempty, we use the notation $\max \mathcal{L}_f V(x)$ (resp., $\min \mathcal{L}_f V(x)$) to denote the largest (resp., smallest) element of $\mathcal{L}_f V(x)$. Furthermore, we adopt the convention $\max \varnothing = -\infty$. Finally, recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ [13, Def. 2.3.4] if, for all $v \in \mathbb{R}^n$, the right directional derivative $V'_+(x, v) \triangleq \lim_{h \to 0^+} \frac{1}{h}[V(x + hv) - V(x)]$ exists and $V'_+(x, v) = V^o(x, v)$. V is called *regular* on $\mathbb{R}^n$ if it is regular at every $x \in \mathbb{R}^n$.

For stating the main stability theorems we assume that all right maximal Filippov solutions to (2.1) exist on $[0, \infty)$. We say that a set $\mathcal{M}$ is *weakly positively invariant* (resp., *strongly positively invariant*) with respect to (2.1) if, for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a right maximal solution (resp., all right maximal solutions) of (2.1) [5, 64]. The set $\mathcal{M} \subseteq \mathbb{R}^q$ is *weakly negatively invariant* if, for every $x \in \mathcal{N}$ and $t \geq 0$, there exist $z \in \mathcal{N}$ and a Filippov solution $\psi(\cdot)$ to (2.1) with $\psi(0) = z$ such that $\psi(t) = x$ and $\psi(\tau) \in \mathcal{N}$ for all $\tau \in [0, t]$. Finally, the set $\mathcal{M} \subseteq \mathbb{R}^q$ is *weakly invariant* if $\mathcal{M}$ is weakly positively invariant as well as weakly negatively invariant.

The next definition introduces the notion of Lyapunov stability, semistability, and asymptotic stability for discontinuous dynamical systems. The adjective "weak" is used in reference to a stability property when the stability property is satisfied by at least one Filippov solution starting from every initial condition in $\mathcal{D}$, whereas "strong" is used when the stability property is satisfied by all Filippov solutions starting from every initial condition in $\mathcal{D}$. In this section, however, we provide strong stability theorems for (2.1) and, hence, we omit the adjective "strong" in the statement of our results.

Definition 2.1.2. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to (2.1). An equilibrium point $x_e \in \mathcal{D}$ of (2.1) is *Lyapunov stable* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every initial condition $x_0 \in \mathcal{B}_\delta(x_e)$ and every Filippov solution $x(t)$ with the initial condition $x(0) = x_0$, $x(t) \in \mathcal{B}_\varepsilon(x_e)$ for all $t \geq 0$. An equilibrium point $x_e \in \mathcal{D}$ of (2.1) is *semistable* if x_e is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing x_e such that, for all initial conditions in $\mathcal{D}_0$, the Filippov solutions of (2.1) converge to a Lyapunov stable equilibrium point. An equilibrium point $x_e \in \mathcal{D}$ of (2.1) is *asymptotically stable* if x_e is Lyapunvov stable and there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e)$, then the Filippov solutions of (2.1) converge to x_e. An equilibrium point $x_e \in \mathcal{D}$ of (2.1) is *exponentially stable* if there exits positive constants α, β, and δ such that if $x_0 \in \mathcal{B}_\delta(x_e)$, then every Filipov solution to (2.1) satisfies $\|x(t)\| \leq \|x_0\| e^{-\beta}$, $t \geq 0$. The system (2.1) is *semistable* (resp., *asymptotically* stable) with respect to $\mathcal{D}$ if every Filippov solution with initial condition in $\mathcal{D}$ converges to a Lyapunov stable equilibrium (resp., the Lyapunov stable equilibrium x_e). Finally, (2.1) is said to be *globally semistable* (resp., *globally asymptotically stable*, *globally exponentially stable*) if (2.1) is semistable (resp., asymptotically stable, exponentially stable) with respect to $\mathbb{R}^n$.

Given an absolutely continuous curve $\gamma : [0, \infty) \to \mathbb{R}^n$, *the positive limit set of* γ is the set $\Omega(\gamma)$ of points $y \in \mathbb{R}^n$ for which there exists an increasing divergent sequence $\{t_i\}_{i=1}^\infty$ satisfying $\lim_{i\to\infty} \gamma(t_i) = y$. We denote the positive limit set of a Filippov solution $\psi(\cdot)$ of (2.1) by $\Omega(\psi)$. The positive limit set of a bounded Filippov solution of (2.1) is nonempty and weakly invariant with respect to (2.1) [24, Lem. 4, p. 130].

2.2. Stability of Discontinuous Dynamical Systems

In this section, we state sufficient conditions for stability of discontinuous dynamical systems. Here, we state the stability theorems for only the local case; the global stability theorems are similar except for the additional assumption of properness on the Lyapunov function and nonrestricting the domain of analysis.

Theorem 2.2.1. ([5,41]) Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (2.1). Let $x_{\rm e}$ be an equilibrium point of $\mathcal{G}$ and let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open and connected set with $x_{\rm e} \in \mathcal{D}$. If $V : \mathcal{D} \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous, and regular function such that $\max \mathcal{L}_f V(x) \leq 0$ (resp., $\max \mathcal{L}_f V(x) < 0$, $x \neq x_{\rm e}$) for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$, then $x_{\rm e}$ is Lyapunov (resp., asymptotically) stable. Finally, if there exists scalars $\alpha, \beta, \gamma > 0$, and $p \geq 1$ such that $V : \mathcal{D} \to \mathbb{R}$ satisfies $\alpha\|x - x_{\rm e}\|^p \leq V(x) \leq \|x - x_{\rm e}\|^p$ and $\max \mathcal{L}_f V(x) \leq -\gamma\|x - x_{\rm e}\|^p$ for almost all $x \in \mathcal{D}$, $x \neq x_{\rm e}$, such that $\mathcal{L}_f V(x) \neq \varnothing$, then $x_{\rm e}$ is exponentially stable.

The next result presents an extenson of the Krasovskii-LaSalle invariant set theorem to discontinuous dynamical systems.

Theorem 2.2.2. ([5,41]) Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (2.1). Let $x_{\rm e}$ be an equilibrium point of $\mathcal{G}$, let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to (2.1) such that $x_{\rm e} \in \mathcal{D}$, and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that, for every $x \in \mathcal{D}$ and every Filippov solution $\psi(\cdot)$ satisfying $\psi(t_0) = x$, there exists a compact subset $\mathcal{D}_{\rm c}$ of $\mathcal{D}$ containing $\psi(t)$ for all $t \geq 0$. Furthermore, assume that $\max \mathcal{L}_f V(x) \leq 0$ for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. Finally, define $\mathcal{R} \triangleq \{x \in \mathcal{D} :\ 0 \in \mathcal{L}_f V(x)\}$ and let $\mathcal{M}$ be the largest weakly positively invariant subset of $\overline{\mathcal{R}} \cap \mathcal{D}$. If $x(t_0) \in \mathcal{D}_{\rm c}$, then

$x(t) \to \mathcal{M}$ as $t \to \infty$. If, alternatively, $\mathcal{R}$ contains no invariant set other than $\{x_e\}$, then the Filippov solution $x(t) \equiv x_e$ of $\mathcal{G}$ is asymptotically stable for all $x_0 \in \mathcal{D}_c$.

Example 2.2.1. Consider a nonsmooth harmonic oscillator with nonsmooth friction given by ([5])

$$\dot{x}_1(t) = -\operatorname{sign}(x_2(t)) - \frac{1}{2}\operatorname{sign}(x_1(t)), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \tag{2.7}$$

$$\dot{x}_2(t) = \operatorname{sign}(x_1(t)), \quad x_2(0) = x_{20}, \tag{2.8}$$

where $\operatorname{sign}(\sigma) \triangleq \frac{\sigma}{|\sigma|}$, $\sigma \neq 0$, and $\operatorname{sign}(0) \triangleq 0$. Next, consider the locally Lipschitz continuous function $V(x) = |x_1| + |x_2|$ and note that

$$\partial V(x) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2: \ x_2 \neq 0, \ x_1 = 0, \\ \overline{\operatorname{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence, with $f(x) = [-\operatorname{sign}(x_2) - \frac{1}{2}\operatorname{sign}(x_1), \operatorname{sign}(x_1)]^{\mathrm{T}}$,

$$\mathcal{L}_f V(x) = \begin{cases} \{-\frac{1}{2}\}, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 = 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2: \ x_2 \neq 0, \ x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) \leq 0$ for almost all $x \in \mathbb{R}^2$ such that $\mathcal{L}_f V(x) \neq \varnothing$, it follows from Theorem 2.2.1 that $(x_1(t), x_2(t)) \equiv (0, 0)$ is Lyapunov stable. △

Example 2.2.2. Consider the harmonic oscillator with Coulomb friction given by ([67])

$$m\ddot{x}(t) + b\operatorname{sign}(\dot{x}(t)) + kx(t) = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad \text{a.e.} \quad t \geq 0, \tag{2.9}$$

or, equivalently,

$$f(x_1, x_2) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_1(t) - \frac{b}{m}\operatorname{sign}(x_2(t)) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$
$$\text{a.e.} \quad t \geq 0,$$

where $m, b, k > 0$. Next, consider the continuously differentiable Lyapunov function $V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$ and note that, for almost all $x \in \mathbb{R}^2$, $\mathcal{L}_f V(x) = \{-b|x_2|\}$, which implies that every Filippov solution of (2.9) approaches the largest weakly positively invariant set in $\overline{\mathcal{R}}$, where $\mathcal{R} \triangleq \{x \in \mathbb{R}^2 : 0 \in \mathcal{L}_f V(x)\}$. Now, since

$$\mathcal{K}[f](x) = \begin{bmatrix} 0 \\ -\frac{k}{m}x_1 - \frac{b}{m}[-1, 1] \end{bmatrix},$$

for $x_2 = 0$, it follows that the largest weakly positively invariant subset of $\overline{\mathcal{R}}$ is $\mathcal{M} = [[-\frac{b}{k}, \frac{b}{k}], 0]^{\mathrm{T}}$. Hence, it follows from Theorem 2.2.2 that $x(t) \to \mathcal{M}$ as $t \to \infty$. $\triangle$

2.3. Static Networks and the Consensus Problem

The consensus problem appears frequently in coordination of multiagent network systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with undirected and directed information flow [40,48,59]. In this dissertation, we use *undirected* and *directed graphs* to represent a static network.

The graph-theoretic notation and terminology we use in this dissertation is standard [27]. Specifically, $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denotes a weighted *directed graph* (or *digraph*) denoting the static network (or static graph) with the set of *nodes* (or *vertices*) $\mathcal{V} = \{1, \ldots, N\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow between agents, and a *weighted adjacency matrix* $\mathcal{A} \in \mathbb{R}^{N \times N}$ such that $\mathcal{A}_{(i,j)} = a_{ij} > 0$, $i, j = 1, \ldots, N$, if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The edge $(j, i) \in \mathcal{E}$ denotes that agent i can obtain information from agent j, but not necessarily vice versa. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$. Note that if the weights a_{ij}, $i, j = 1, \ldots, N$, are not relevant, then a_{ij} is set to 1 for all $(j, i) \in \mathcal{E}$. In this case, $\mathcal{A}$ is called a *normalized adjacency matrix*.

Every edge $\ell \in \mathcal{E}$ corresponds to an ordered pair of vertices $(i, j) \in \mathcal{V} \times \mathcal{V}$, where i and j are the *initial* and *terminal* vertices of the edge ℓ. In this case, ℓ is *incident into* j and *incident out of* i. Finally, we say that $\mathfrak{G}$ is *strongly* (resp., *weakly*) *connected* if for every ordered pair of vertices (i, j), $i \neq j$, there exists a *directed* (resp., *undirected*) *path*, that is, a directed (resp., undirected) sequence of arcs, leading from i to j.

The *in-neighbors* and *out-neighbors* of node i are, respectively, defined as $\mathcal{N}_{\text{in}}(i) \triangleq \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_{\text{out}}(i) \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The *in-degree* $\deg_{\text{in}}(i)$ of node i is the number of edges incident into i and the *out-degree* $\deg_{\text{out}}(i)$ of node i is the number of edges incident out of i, that is, $\deg_{\text{in}}(i) \triangleq \sum_{j=1}^{N} a_{ji}$ and $\deg_{\text{out}}(i) \triangleq \sum_{j=1}^{N} a_{ij}$. We say that the node i of a digraph $\mathfrak{G}$ is *balanced* if $\deg_{\text{in}}(i) = \deg_{\text{out}}(i)$, and a graph $\mathfrak{G}$ is called *balanced* if all of its nodes are balanced, that is, $\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$, $i = 1, \ldots, N$. Furthermore, we define the *graph Laplacian* and *Perron matrix* of $\mathfrak{G}$ as $\mathcal{L} \triangleq \Delta - \mathcal{A}$ and $\mathcal{P} \triangleq I - \varepsilon\mathcal{L}$, respectively, where $\varepsilon > 0$ and $\Delta \triangleq \text{diag}[\deg_{\text{in}}(1), \ldots, \deg_{\text{in}}(N)]$.

A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$ is a directed graph for which the arc set is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\text{T}}$. In this case, $\mathcal{N}_{\text{in}}(i) = \mathcal{N}_{\text{out}}(i) \triangleq \mathcal{N}(i)$ and $\deg_{\text{in}}(i) = \deg_{\text{out}}(i) \triangleq \deg(i)$, $i = 1, \ldots, N$. Furthermore, in this case we say that $\mathfrak{G}$ is *connected* if for every ordered pair of vertices (i, j), $i \neq j$, there exists a *path*, that is, a sequence of arcs, leading from i to j. A graph is *all-to-all connected* if every node of $\mathfrak{G}$ is connected to every other node of $\mathfrak{G}$. Finally, we denote the value of the node $i \in \{1, \ldots, N\}$ at time t (resp., time step k) by $x_i(t) \in \mathbb{R}^n$ (resp., $x_i(k) \in \mathbb{R}^n$).

In light of the above definitions, the consensus problem involves the design of a dynamic algorithm that guarantees system state equipartition [48, 59], that is,

$\lim_{t\to\infty} x_i(t) = q \in \mathbb{R}^n$ for $i = 1, \ldots, N$. This problem can be characterized as a network involving trajectories of a multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N, \tag{2.10}$$

$$u_i(t) = \sum_{j \in \mathcal{N}_{\rm in}(i)} \phi_{ij}(x_i(t), x_j(t)), \quad i = 1, \ldots, N, \tag{2.11}$$

where $\phi_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, N$, are locally Lipschitz continuous. Here, $x_i(t)$, $t \geq 0$, represents an *information state* and $u_i(t)$, $t \geq 0$, repsresents an *information control input* with a distributed consensus algorithm involving neighbor-to-neighbor interaction between agents. In this dissertation, we consider continuous-time distributed consensus algorithms resulting in closed-loop systems of the form ([59])

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_{\rm in}(i)} (x_j(t) - x_i(t)), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N, \tag{2.12}$$

as well as discrete-time distributed consensus algorithms resulting in closed-loop systems of the form ([48])

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_{\rm in}(i)} (x_j(k) - x_i(k)), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, N, \tag{2.13}$$

where $\varepsilon > 0$.

Chapter 3

Universal Feedback Controllers for Discontinuous Systems

3.1. Introduction

The consideration of nonsmooth Lyapunov functions for proving stability of feedback discontinuous systems is an important extension to classical stability theory since, as shown in [67], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein [3] to show the existence of a feedback stabilizing controller. A constructive feedback control law based on smooth control Lyapunov functions was given in [68]. Even though a stabilizing continuous feedback controller guarantees the existence of a smooth control Lyapunov function, many systems that possess smooth control Lyapunov functions do not necessarily admit a continuous stabilizing feedback controller [3, 61]. However, as shown in [61], the existence of a control Lyapunov function allows for the design of a stabilizing feedback controller that admits Filippov and Krasovskii closed-loop system solutions. In this chapter, we use the results of [60, 61] to develop a constructive universal feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients [13] and set-valued Lie derivatives [5].

Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = F(x(t), u(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{3.1}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [24] with respect to x, continuous with respect to u, and admits an equilibrium point at $x_e \in \mathcal{D}$ for some $u_e \in U$; that is, $F(x_e, u_e) = 0$. The control $u(\cdot)$ in (3.1) is restricted to the class of *admissible* controls consisting of measurable and locally essentially bounded functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$. For each value $u \in U$, we define the function F_u by $F_u(x) = F(x, u)$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(x_e) = u_e$ is called a *control law*. If $u(t) = \phi(x(t))$, where ϕ is a control law and $x(t)$ satisfies (3.1), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, the *closed-loop system* is given by

$$\dot{x}(t) = F(x(t), \phi(x(t))), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0. \tag{3.2}$$

Analogous to the open-loop case, we define the function F_ϕ by $F_\phi(x) = F(x, \phi(x))$. Note that an arc $x(\cdot)$ (i.e., an absolutely continuous function from $[t_0, t]$ to $\mathcal{D}$) satisfies (3.1) for an admissible control $u(t) \in U$ if and only if [24, p. 152]

$$\dot{x}(t) \in \mathcal{F}(x(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{3.3}$$

where $\mathcal{F}(x) \triangleq F(x, U)$, that is, $\mathcal{F}(x) \triangleq \{F(x, u) : u \in U\}$.

Here $\mathcal{F} : \mathcal{D} \to 2^{\mathbb{R}^n}$ is a *set-valued map* that assigns sets to points. The set $\mathcal{F}(x)$ captures all of the directions in $\mathbb{R}^n$ that can be generated at x with inputs $u = u(t) \in U$. The inputs $u(\cdot)$ can be selected as either $u : [t_0, \infty) \to U$ or $u : \mathcal{D} \to U$.

We assume that $\mathcal{F}(x)$ is an upper semicontinuous, nonempty, convex, and compact set for all $x \in \mathbb{R}^n$. That is, for every $x \in \mathcal{D}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \mathbb{R}^n$ satisfying $\|z - x\| \leq \delta$, $\mathcal{F}(z) \subseteq \mathcal{F}(x) + \overline{\mathcal{B}}_\varepsilon(0)$. This assumption is mainly used to guarantee the existence of Filippov solutions to (3.2) [24].

An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^n$ is said to be a Filippov solution [24] of (3.2) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[F_\phi](x(t)), \quad \text{a.e.} \quad t \in [t_0, \tau], \tag{3.4}$$

where the Filippov set-valued map $\mathcal{K}[F_\phi] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\mathcal{K}[F_\phi](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{co}}\{F_\phi(\mathcal{B}_\delta(x) \backslash \mathcal{S})\}, \quad x \in \mathcal{D}, \tag{3.5}$$

$\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, "$\overline{\text{co}}$" denotes convex closure, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero.[2] Note that since F is locally essentially bounded, $\mathcal{K}[F_\phi](\cdot)$ is upper semicontinuous and has nonempty, compact, and convex values.

3.2. Nonsmooth Control Lyapunov Functions

In this section, we consider a feedback control problem and introduce the notion of *control Lyapunov functions* for discontinuous dynamical systems. Furthermore, using the concept of control Lyapunov functions we provide necessary and sufficient conditions for stabilization of discontinuous nonlinear dynamical systems. To address the problem of control Lyapunov functions for discontinuous dynamical systems, let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set and let $U \subseteq \mathbb{R}^m$, where $0 \in \mathcal{D}$ and $0 \in U$. Next, consider the controlled nonlinear discontinuous dynamical system (3.1), where $u(\cdot)$ is restricted

[2] Alternatively, we can consider Krasovskii solutions of (3.2) wherein the possible misbehavior of the derivative of the state on null measure sets is not ignored; that is, $\mathcal{K}[F_\phi](x)$ is replaced with $\mathcal{K}[F_\phi](x) = \bigcap_{\delta>0} \overline{\text{co}}\{F_\phi(\mathcal{B}_\delta(x))\}$ and where F_ϕ is assumed to be locally bounded.

to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$ for almost all $t \geq 0$ and the constraint set U is given. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the closed-loop dynamical system is given by (3.2).

The following definitions are required for stating the main result of this section.

Definition 3.2.1. Let $\phi : \mathcal{D} \to U$ be a measurable mapping on $\mathcal{D}\backslash\{0\}$ with $\phi(0) = 0$. Then (3.1) is *feedback asymptotically stabilizable* if the zero Filippov solution $x(t) \equiv 0$ of the closed-loop discontinuous nonlinear dynamical system (3.2) is asymptotically stable.

Definition 3.2.2. Consider the controlled discontinuous nonlinear dynamical system given by (3.1). A locally Lipschitz continuous, regular, and positive-definite function $V : \mathcal{D} \to \mathbb{R}$ satisfying

$$\inf_{u \in U} [\max \mathcal{L}_{F_u} V(x)] < 0, \quad \text{a.e.} \quad x \in \mathcal{D} \setminus \{0\}, \tag{3.6}$$

is called a *control Lyapunov function.*

Note that if (3.6) holds, then there exists a feedback control law $\phi : \mathcal{D} \to U$ such that $\max \mathcal{L}_{F_\phi} V(x) < 0$, $x \in \mathcal{D}$, $x \neq 0$, and hence, Theorem 2.2.1 with $f(x) = F_\phi(x) = F(x, \phi(x))$ implies that if there exists a control Lyapunov function for the discontinuous nonlinear dynamical system (3.1), then there exists a feedback control law $\phi(x)$ such that the zero Filippov solution $x(t) \equiv 0$ of the closed-loop system (3.2) is strongly asymptotically stable. Conversely, if there exists a feedback control law $u = \phi(x)$ such that the zero Filippov solution $x(t) \equiv 0$ of the discontinuous nonlinear dynamical system (3.1) is strongly asymptotically stable, then, since $\mathcal{L}_{F_\phi} V(x) \subseteq \{p^{\mathrm{T}} v : p^{\mathrm{T}} \in \partial V(x) \text{ and } v \in \mathcal{K}[F_\phi](x)\}$, it follows from Theorem 2.7 of [61] that there exists a locally Lipschitz continuous, regular, and positive-definite function $V : \mathcal{D} \to \mathbb{R}$ such that $\max \mathcal{L}_{F_\phi} V(x) < 0$ for all nonzero $x \in \mathcal{D}$ or, equivalently, there exists a control Lyapunov function for the discontinous nonlinear dynamical system (3.1). Hence, a given discontinuous dynamical system of the form (3.1) is strongly feedback asymptotically stabilizable if and only if there exists a control Lyapunov function satisfying (3.6). Finally, in the case where $\mathcal{D} = \mathbb{R}^n$ and $U = \mathbb{R}^m$ the zero Filippov solution $x(t) \equiv 0$ to (3.1) is globally strongly asymptotically stabilizable if and only if $V(x) \to \infty$ as $||x|| \to \infty$.

Next, we consider the special case of discontinuous nonlinear systems affine in the control, and we construct state feedback controllers that globally asymptotically stabilize the zero Filippov solution of the discontinuous nonlinear dynamical system under the assumption that the system has a radially unbounded control Lyapunov function. Specifically, we consider discontinuous nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{3.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. We assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. Note that (3.7) is a special case of (3.1) with $F(x, u) = f(x) + G(x)u$. We use the notation $f + Gu$ to denote the function $F_u(x) = f(x) + G(x)u$ for each $u \in \mathbb{R}^m$.

Note that (3.7) includes piecewise continuous dynamical systems as well as switched dynamical systems as special cases. For example, if $f(\cdot)$ and $G(\cdot)$ are piecewise continuous, then (3.7) can be represented as a differential inclusion involving Filippov set-valued maps of piecewise-continuous vector fields given by $\mathcal{K}[f](x) = \overline{\text{co}}\{\lim_{i\to\infty} f(x_i) : x_i \to x, x_i \notin \mathcal{S}_f\}$, where $\mathcal{S}_f$ has measure zero and denotes the set of points where f is discontinuous [56], and similarly for $G(\cdot)$. Here, we assume that $\mathcal{K}[f](\cdot)$ has at least one equilibrium point so that, without loss of generality, $0 \in \mathcal{K}[f](0)$.

Next, define

$$\mathcal{L}_G V(x) \triangleq \{q \in \mathbb{R}^{1\times m} : \text{there exists } v \in \mathfrak{G}(x) \text{ such that } p^{\mathrm{T}} v = q \text{ for all } p^{\mathrm{T}} \in \partial V(x)\},$$

where $\mathfrak{G}(x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\{G(\mathcal{B}_\delta(x)\backslash\mathcal{S})\}$, $x \in \mathbb{R}^n$, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. Finally, we assume that the set $\mathcal{L}_G V(x)$ is single-valued[3] for almost all $x \in \mathbb{R}^n$, and that $\mathcal{L}_G V(x) \neq \varnothing$ at all other points x.

Theorem 3.2.1. Consider the controlled discontinuous nonlinear dynamical system given by (3.7). Then a locally Lipschitz continuous, regular, positive-definite, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ is a control Lyapunov function for (3.7) if and only if

$$\max \mathcal{L}_f V(x) < 0, \quad \text{a.e.} \quad x \in \mathcal{R}, \tag{3.8}$$

where $\mathcal{R} \triangleq \{x \in \mathbb{R}^n \setminus \{0\} : \mathcal{L}_G V(x) = 0\}$.

Proof. Sufficiency is a direct consequence of the definition of a control Lyapunov function and the sum rule for computing the generalized gradient of locally Lipschitz continuous functions [56]. Specifically, for systems of the form (3.7), note that $\mathcal{L}_{f+Gu} V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_G V(x) u$ for almost all x and all u, and hence,

$$\inf_{u \in U} [\max \mathcal{L}_f V(x) + \mathcal{L}_G V(x) u] = -\infty$$

[3]The assumption that $\mathcal{L}_G V(x)$ is single-valued is necessary. Specifically, as will be seen later, the requirement that there exists $\overline{z} \in \mathcal{L}_G V(x)$ such that, for all $u \in \mathbb{R}^m$, $\max[\mathcal{L}_G V(x) u] = \overline{z} u$ holds if and only if $\mathcal{L}_G V(x)$ is a singleton. To see this, let $q, r \in \mathcal{L}_G V(x)$, with $q \neq r$, and assume, *ad absurdum*, that the required $\overline{z}$ exists. Then, either $q - \overline{z} \neq 0$ or $r - \overline{z} \neq 0$. Assume $q - \overline{z} \neq 0$ and let $u^{\mathrm{T}} = q - \overline{z}$. Then, $qu - \overline{z}u = (q - \overline{z})u = (q - \overline{z})(q - \overline{z})^{\mathrm{T}} = \|q - \overline{z}\|_2^2 > 0$. Hence, $qu > \overline{z}u$, which leads to a contradiction.

when $x \notin \mathcal{R}$ and $x \neq 0$, whereas $\inf_{u\in U}[\max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)u] < 0$ for almost all $x \in \mathcal{R}$. Hence, (3.8) implies (3.6) with $F_u(x) = f(x) + G(x)u$.

To prove necessity suppose, *ad absurdum*, that $V(\cdot)$ is a control Lyapunov function and (3.8) does not hold. In this case, there exists a set $\mathcal{M} \subseteq \mathcal{R}$ of positive measure such that $\max \mathcal{L}_f V(x) \geq 0$ for all $x \in \mathcal{M}$. Let $x \in \mathcal{M}$ and let $\alpha \in \mathcal{L}_f V(x) \cap [0, \infty)$. From the definition of a control Lyapunov function, x is such that there exists u such that $\max \mathcal{L}_{f+Gu} V(x) < 0$ and, by the sum rule for generalized gradients, the inclusion $\mathcal{L}_f V(x) \subseteq \mathcal{L}_{f+Gu} V(x) + \mathcal{L}_{-Gu} V(x)$ is satisfied (since the sum rule holds for almost all x). Now, since $x \in \mathcal{M}$, we have $\mathcal{L}_{-Gu} V(x) = -\mathcal{L}_{Gu} V(x) \subseteq -\mathcal{L}_G V(x)u \subseteq \{0\}$. Hence, there exists a nonnegative $\alpha \in \mathcal{L}_{f+Gu} V(x)$, which is a contradiction. This proves the theorem. $\square$

It follows from Theorem 3.2.1 that the zero Filippov solution $x(t) \equiv 0$ of a discontinuous nonlinear affine system of the form (3.7) is globally strongly feedback asymptotically stabilizable if and only if there exists a locally Lipschitz continuous, regular, positive-definite, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying (3.8). Hence, Theorem 3.2.1 provides necessary and sufficient conditions for discontinuous nonlinear system stabilization.

Next, using Theorem 3.2.1 we *construct* an explicit feedback control law that is a function of the control Lyapunov function $V(\cdot)$. Specifically, consider the feedback control law given by

$$\phi(x) = \begin{cases} -\left(c_0 + \frac{\alpha(x)+\sqrt{\alpha^2(x)+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0, \end{cases} \tag{3.9}$$

where $\alpha(x) \triangleq \max \mathcal{L}_f V(x)$, $\beta(x) \triangleq (\mathcal{L}_G V(x))^{\mathrm{T}}$, and $c_0 \geq 0$ is a constant. In this case, the control Lyapunov function $V(\cdot)$ of (3.7) is a Lyapunov function for the closed-loop system (3.7) with $u = \phi(x)$, where $\phi(x)$ is given by (3.9). To see this, recall that using the sum rule for computing the generalized gradient of locally Lipschitz continuous functions [56] it follows that $\mathcal{L}_{f+Gu}V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_{Gu}V(x)$ for almost all $x \in \mathbb{R}^n$. Now, Theorem 3.2.1 gives

$$\begin{aligned}
\max \mathcal{L}_{F_\phi} V(x) &= \max \mathcal{L}_{f+G\phi} \\
&\leq \max [\mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x)] \\
&= \max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x) \\
&= \alpha(x) + \beta^T(x)\phi(x) \\
&= \begin{cases} -c_0\beta^{\mathrm{T}}(x)\beta(x) - \sqrt{\alpha^2(x) + (\beta^{\mathrm{T}}(x)\beta(x))^2}, & \beta(x) \neq 0, \\ \alpha(x), & \beta(x) = 0, \end{cases} \\
&< 0, \quad x \in \mathbb{R}^n, \quad \text{a.e.} \quad x \neq 0, \qquad (3.10)
\end{aligned}$$

which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (3.7), and hence, by Theorem 2.2.1, guaranteeing global strong asymptotic stability with $u = \phi(x)$ given by (3.9).

3.3. Illustrative Numerical Examples

In this section, we present several numerical examples to illustrate the utility of the proposed feedback control law.

Example 3.3.1. Consider a controlled nonsmooth harmonic oscillator with nonsmooth friction given by ([5])

$$\dot{x}_1(t) = -\operatorname{sign}(x_2(t)) - \frac{1}{2}\operatorname{sign}(x_1(t)), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \qquad (3.11)$$

$$\dot{x}_2(t) = \operatorname{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20}, \qquad (3.12)$$

where $\operatorname{sign}(\sigma) \triangleq \frac{\sigma}{|\sigma|}$, $\sigma \neq 0$, and $\operatorname{sign}(0) \triangleq 0$. Next, consider the locally Lipschitz continuous function $V(x) = |x_1| + |x_2|$ and note that

$$\partial V(x) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : \ x_1 \neq 0, \ x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2 : \ x_1 \neq 0, \ x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : \ x_2 \neq 0, \ x_1 = 0, \\ \overline{\operatorname{co}}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence, with $f(x) = [-\operatorname{sign}(x_2) - \frac{1}{2}\operatorname{sign}(x_1), \operatorname{sign}(x_1)]^{\mathrm{T}}$ and $G = [0, 1]^{\mathrm{T}}$,

$$\mathcal{L}_f V(x) = \begin{cases} \{-\frac{1}{2}\}, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 = 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2: \ x_2 \neq 0, \ x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

and

$$\mathcal{L}_G V(x) = \begin{cases} \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2: \ x_1 \neq 0, \ x_2 = 0, \\ \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2: \ x_2 \neq 0, \ x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{R}$, where $\mathcal{R} = \{x \in \mathbb{R}^2 \setminus \{0\} : \ \mathcal{L}_G V(x) = 0\}$, it follows from Theorem 3.2.1 that $V(x) = |x_1| + |x_2|$ is a control Lyapunov function for (3.11) and (3.12).

Next, note that it follows from (3.9) that

$$\begin{aligned} \phi(x) &= \begin{cases} -\left(c_0 + \dfrac{-\frac{1}{2} + \sqrt{\frac{1}{4} + \operatorname{sign}^4(x_2)}}{\operatorname{sign}^2(x_2)}\right) \operatorname{sign}(x_2), & \operatorname{sign}(x_2) \neq 0, \\ 0, & \operatorname{sign}(x_2) = 0, \end{cases} \\ &= \begin{cases} -\left(c_0 + \dfrac{\sqrt{5} - 1}{2}\right) \operatorname{sign}(x_2), & \operatorname{sign}(x_2) \neq 0, \\ 0, & \operatorname{sign}(x_2) = 0, \end{cases} \end{aligned} \tag{3.13}$$

where $c_0 \geq 0$, and hence, since $\mathcal{L}_{f+G\phi} V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x)$ for almost all x,

$$\max \mathcal{L}_{f+G\phi} V(x) \ \leq \ -\left(c_0 + \frac{\sqrt{5}}{2}\right) \ < \ 0, \quad \operatorname{sign}(x_2) \neq 0.$$

Now, it follows from Theorem 2.2.1 that (3.13) is a globally strongly stabilizing feedback controller. Figures 3.1 and 3.2 show the phase portraits of the open-loop ($u(t) \equiv 0$) and closed-loop nonsmooth harmonic oscillator with $c = 0$, respectively. Finally, Figures 3.3 and 3.4 show the state trajectories and the control trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\mathrm{T}}$ and $c = 0$. $\triangle$

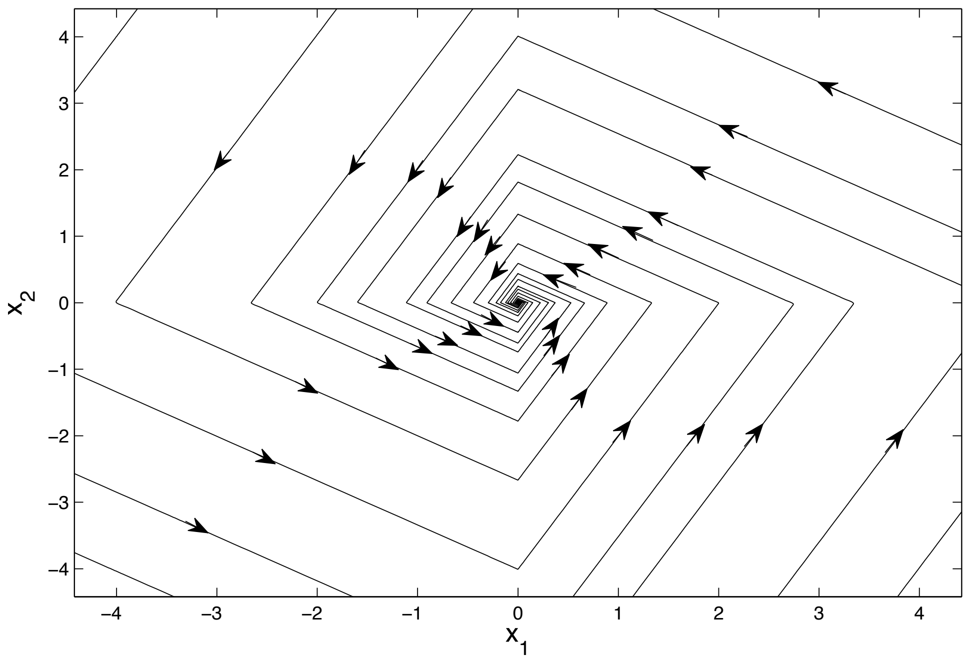

Figure 3.1. Phase portrait of the open-loop nonsmooth harmonic oscillator.

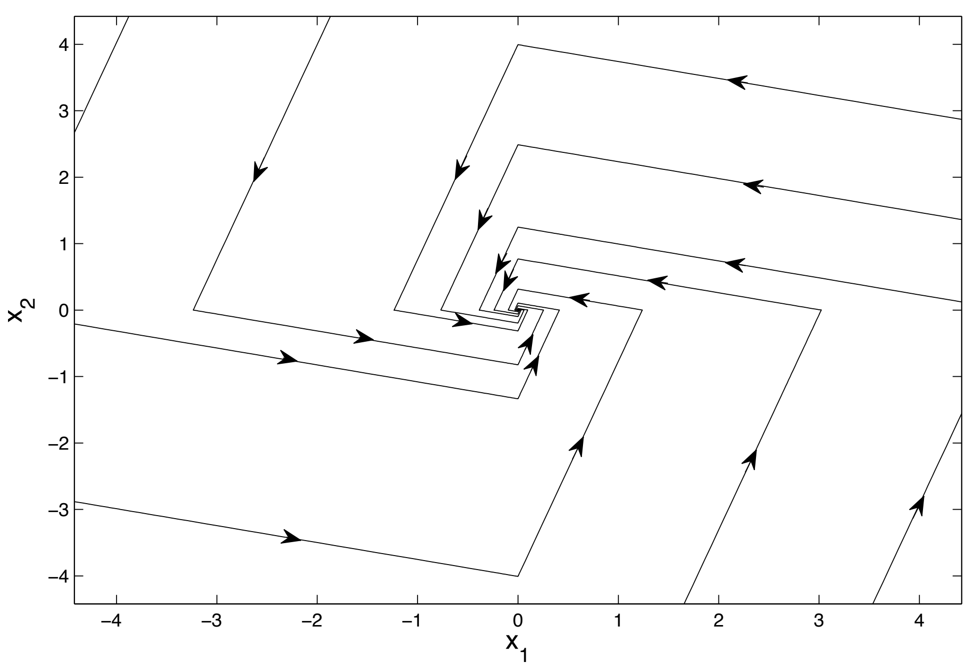

Figure 3.2. Phase portrait of the closed-loop nonsmooth harmonic oscillator.

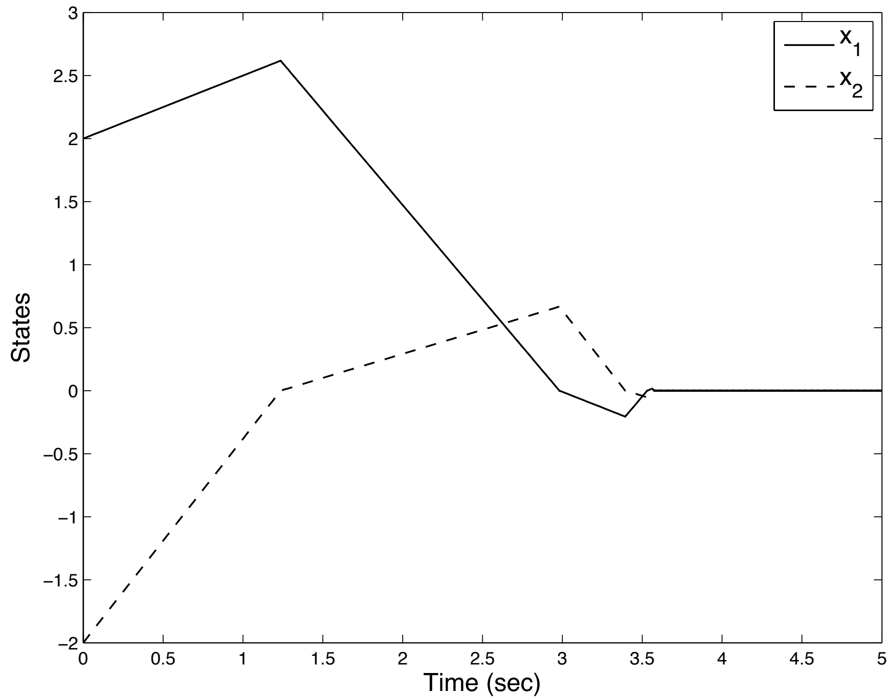

Figure 3.3. State trajectories of the closed-loop system versus time.

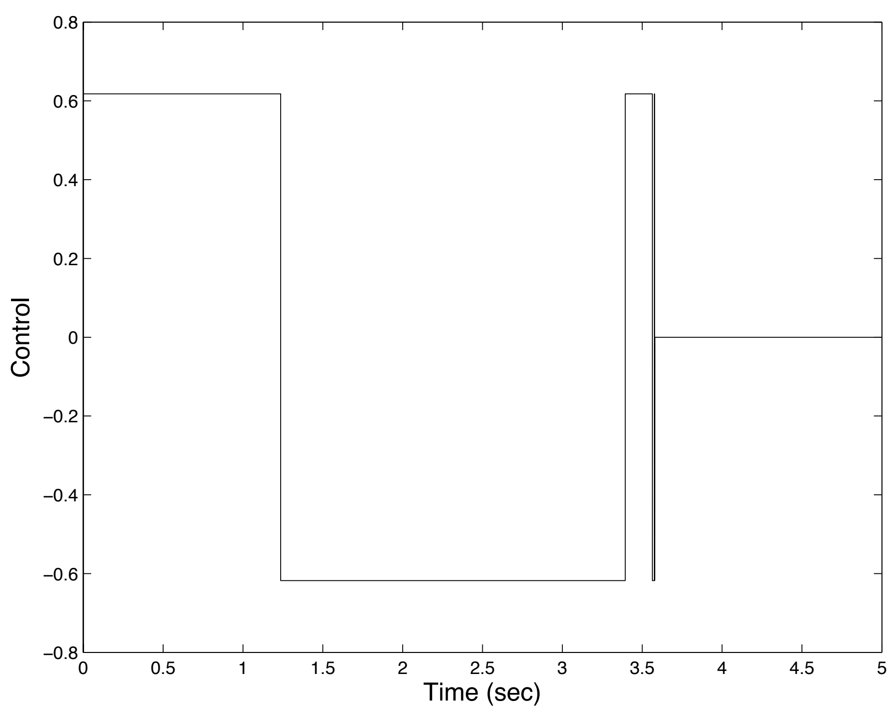

Figure 3.4. Control trajectories of the closed-loop system versus time.

Example 3.3.2. Consider the controlled dynamical system $\mathcal{G}$ given by (3.7), where $x = [x_1, x_2]^{\mathrm{T}}$, $u = [u_1, u_2]^{\mathrm{T}}$,

$$f(x) = \begin{bmatrix} |x_1|(-x_1 + |x_2|) \\ x_2(-x_1 - |x_2|) \end{bmatrix}, \quad G(x) = \begin{bmatrix} |x_1| & 0 \\ 0 & x_2 \end{bmatrix}.$$

Next, consider the locally Lipschitz continuous function $V(x) = 2|x_1| + 2|x_2|$ and note that

$$\partial V(x) = \begin{cases} \{2\,\mathrm{sign}(x_1)\} \times \{2\,\mathrm{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_1 \neq 0, \ x_2 \neq 0, \\ \{2\,\mathrm{sign}(x_1)\} \times [-2, 2], & (x_1, x_2) \in \mathbb{R}^2: \ \ x_1 \neq 0, \ x_2 = 0, \\ [-2, 2] \times \{2\,\mathrm{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_2 \neq 0, \ x_1 = 0, \\ \overline{\mathrm{co}}\{(2, 2), (-2, 2), (-2, -2), (2, -2)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V(x) = \begin{cases} \{-2x_1^2 - 2x_2^2\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_1 \neq 0, \ x_2 \neq 0, \\ \{-2x_1^2\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_1 \neq 0, \ x_2 = 0, \\ \{-2x_2^2\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_2 \neq 0, \ x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

and

$$\mathcal{L}_G V(x) = \begin{cases} \{(2x_1, 2|x_2|)\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_1 \neq 0, \ x_2 \neq 0, \\ \{(2x_1, 0)\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_1 \neq 0, \ x_2 = 0, \\ \{(0, 2|x_2|)\}, & (x_1, x_2) \in \mathbb{R}^2: \ \ x_2 \neq 0, \ x_1 = 0, \\ \{(0, 0)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{R}$, where $\mathcal{R} = \{x \in \mathbb{R}^2 \setminus \{0\}: \ \mathcal{L}_G V(x) = 0\}$, it follows from Theorem 3.2.1 that $V(x) = 2|x_1| + 2|x_2|$ is a control Lyapunov function.

Setting $\alpha(x) = \max \mathcal{L}_f V(x)$ and $\beta(x) = (\mathcal{L}_G V(x))^{\mathrm{T}}$, it follows that $\beta(x)\beta^T(x) = 4(x_1^2 + x_2^2)$ and $\alpha^2(x) + (\beta^T(x)\beta(x))^2 = 4(x_1^2 + x_2^2)^2 + 16(x_1^4 + x_2^4 + 2x_1^2 x_2^2) = 20(x_1^4 + x_2^4) + 40x_1^2 x_2^2 = 20(x_1^2 + x_2^2)^2$, and hence, (3.9) gives

$$\phi(x) = \begin{cases} -\left(c_0 + (\sqrt{5} - 1)\right) \begin{bmatrix} x_1 \\ |x_2| \end{bmatrix}, & (x_1, x_2) \neq (0, 0), \\ 0, & (x_1, x_2) = (0, 0), \end{cases} \tag{3.14}$$

where $c_0 \geq 0$. Thus, $\max \mathcal{L}_{f+G\phi} V(x) \leq -|x|^2$ for all $x \neq 0$. Now, it follows from Theorem 2.2.1 that (3.14) is a globally strongly stabilizing feedback controller. Figures 3.5 and 3.6 show the phase portraits of the open-loop ($u(t) \equiv 0$) and closed-loop system with $c = 50$, respectively. Finally, Figures 3.7 and 3.8 show the state trajectories

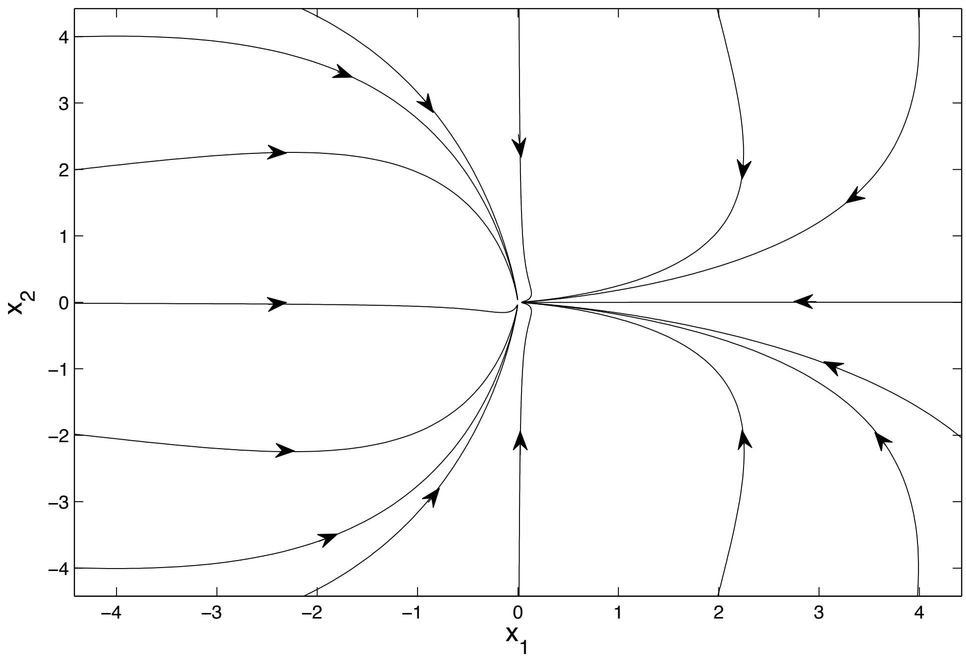

Figure 3.5. Phase portrait of the open-loop system.

and the control trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\mathrm{T}}$ and $c = 50$. $\triangle$

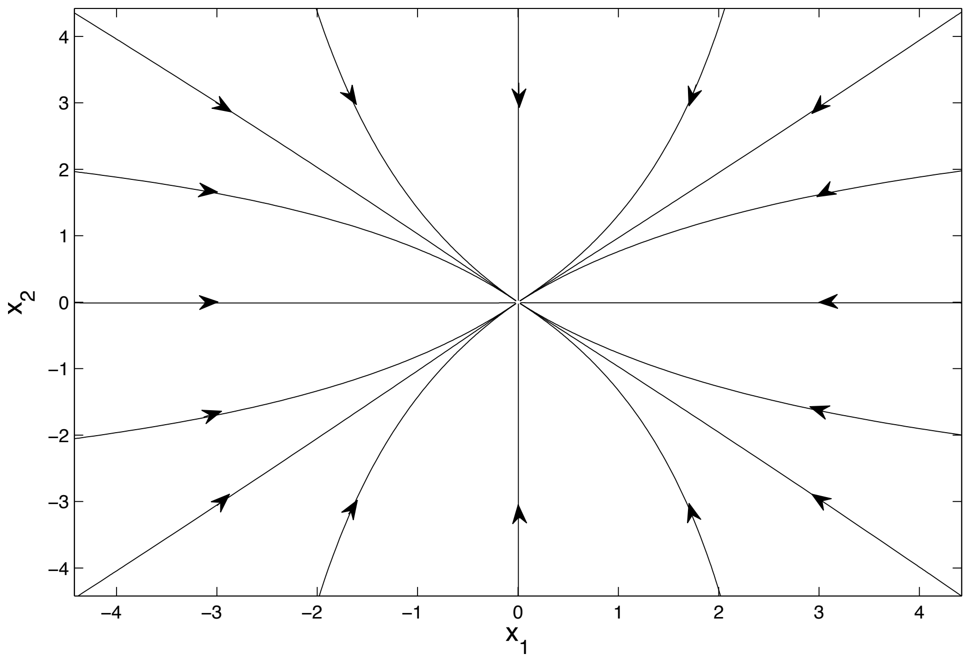

Figure 3.6. Phase portrait of the closed-loop system.

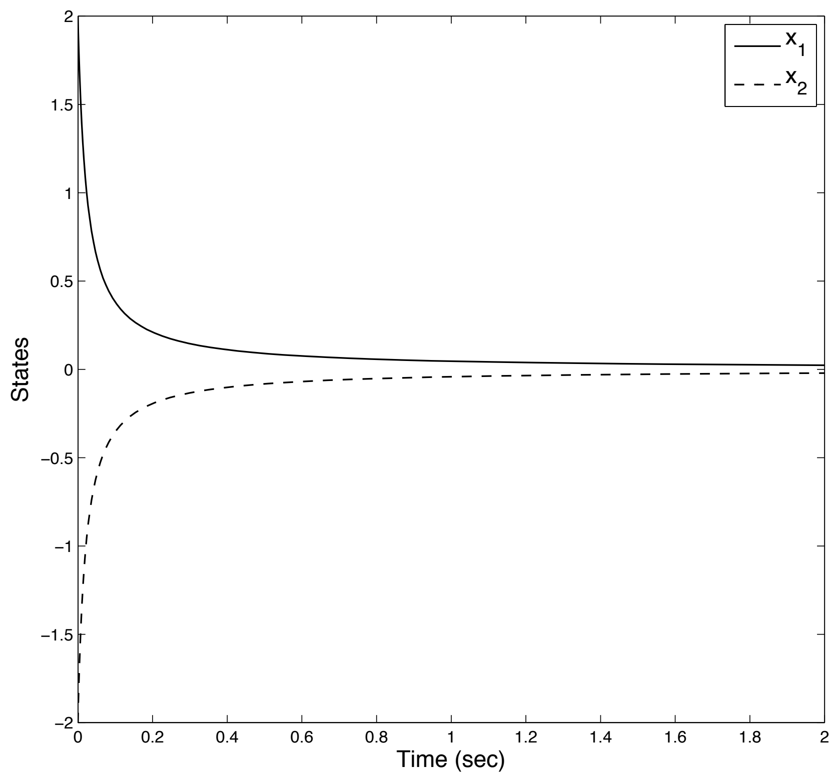

Figure 3.7. State trajectories of the closed-loop system versus time.

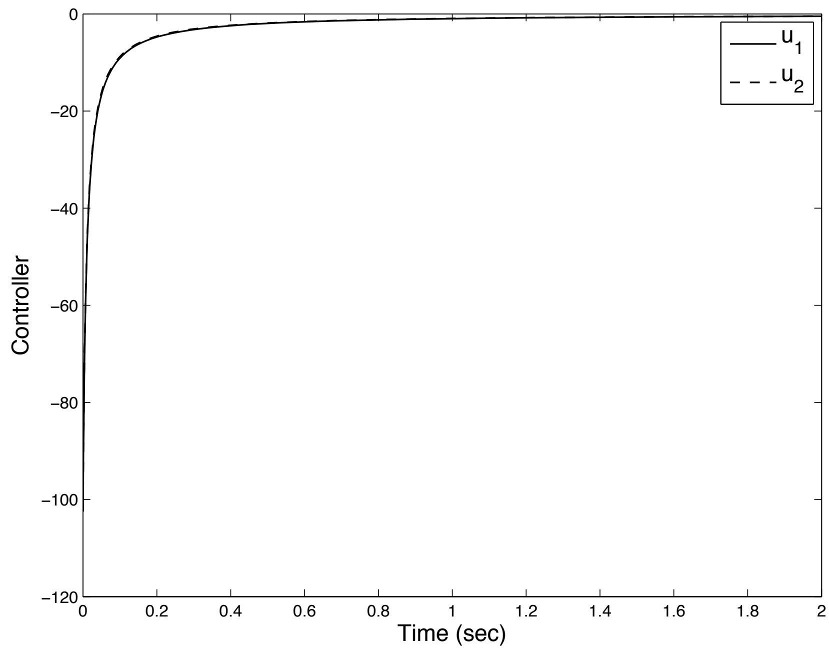

Figure 3.8. Control trajectories of the closed-loop system versus time.

Chapter 4

Dissipativity Theory for Discontinuous Systems

4.1. Introduction

Dissipativity theory is a system-theoretic concept that provides a powerful framework for the analysis and control design of dynamical systems based on generalized system energy considerations. The key foundation in developing dissipativity theory for nonlinear dynamical systems with continuously differentiable flows was presented by Willems [74, 75] in his seminal two-part paper on dissipative dynamical systems. Dissipativity theory along with Lyapunov stability theory for feedback interconnections of dissipative systems has been extensively developed for continuous dynamical systems possessing continuously differentiable flows [31]. In light of the fact that energy notions involving conservation, dissipation, and transport also arise naturally for discontinuous systems, it seems natural that dissipativity theory can play a key role in the analysis and control design of discontinuous dynamical systems. Specifically, dissipativity theory can be used to analyze robust stability of discontinuous dynamical systems. Moreover, it can be applied to design discontinuous feedback controllers that add dissipation and guarantee stability robustness allowing discontinuous stabilization to be understood in physical terms.

In [35], the authors extend the notion of dissipativity theory to impulsive and hybrid dynamical systems possessing left-continuous flows using generalized storage functions and hybrid supply rates. The overall approach provides an interpretation of a generalized energy balance for impulsive and hybrid dynamical systems in terms of the stored or accumulated system generalized energy, the dissipated energy over the continuous-time dynamics, and the dissipated energy at the resetting instants. Extensions of dissipativity theory to vector dissipativity notions using vector storage functions and vector supply rates for analyzing large-scale interconnected systems are considered in [37]. More recently passivity theory for switched dynamical systems described by a family of subsystems parameterized by a finite index set are discussed in [36, 78–80].

In this chapter, we extend the results of [36] to develop dissipativity notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves.

Finally, using generalized Clarke gradients of locally Lipschitz continuous storage functions, we develop extended Kalman-Yakubovich-Popov conditions for discontinuous systems. In addition, using the concepts of dissipativity, we construct nonsmooth Lyapunov functions for discontinuous feedback systems as well as provide general stability criteria for feedback interconnections of discontinuous dynamical systems. The consideration of nonsmooth Lyapunov functions for proving stability of feedback interconnections of discontinuous systems is an important extension to classical stability theory of dissipative feedback systems since, as shown in [67], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory.

4.2. Dissipative Discontinuous Dynamical Systems

In this section, we extend the notion of classical dissipativity [74, 75] of dynamical systems with continuously differentiable flows to discontinuous systems. Specifically, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = F(x(t), u(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{4.1}$$

$$y(t) = H(x(t), u(t)), \tag{4.2}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [24] with respect to x, continuous with respect to u, admits an equilibrium point at $x_e \in \mathcal{D}$ for some $u_e \in U$; that is, $F(x_e, u_e) = 0$, and $H : \mathcal{D} \times U \to \mathbb{R}^l$. The following definition is needed for the main results of this section.

Definition 4.2.1. *i*) The discontinuous dynamical system $\mathcal{G}$ given by (4.1) and (4.2) is *weakly* (resp., *strongly*) *dissipative with respect to the* (locally Lebesgue integrable) *supply rate* $s : U \times Y \to \mathbb{R}$ if there exists a locally Lipschitz continuous, regular, and nonnegative definite *storage function* $V_s : \mathcal{D} \to \mathbb{R}$, such that $V_s(0) = 0$ and the dissipation inequality

$$V_s(x(t)) \leq V_s(x(t_0)) + \int_{t_0}^{t} s(u(\sigma), y(\sigma))\mathrm{d}\sigma, \quad t \geq t_0, \tag{4.3}$$

is satisfied for at least one (resp., every) Filippov solution $x(t)$, $t \geq t_0$, of $\mathcal{G}$ with $u(t) \in U$.

ii) The discontinuous dynamical system $\mathcal{G}$ given by (4.1) and (4.2) is *weakly* (resp., *strongly*) *exponentially dissipative with respect to the* (locally Lebesgue integrable) *sypply rate* $s : U \times Y \to \mathbb{R}$ if there exist a locally Lipschitz continuous, regular, and nonnegative storage function $V_s : \mathcal{D} \to \mathbb{R}$ and a scalar $\varepsilon > 0$ such that $V_s(0) = 0$ and the dissipation inequality

$$e^{\varepsilon t}V_{\rm s}(x(t)) \leq e^{\varepsilon t_0}V_{\rm s}(x(t_0)) + \int_{t_0}^{t} e^{\varepsilon\sigma}s(u(\sigma),y(\sigma)){\rm d}\sigma, \quad t \geq t_0, \tag{4.4}$$

is satisfied for one (resp., every) Filippov solution $x(t)$, $t \geq 0$, of $\mathcal{G}$ with $u(t) \in U$.

iii) The discontinuous dynamical system $\mathcal{G}$ given by (4.1) and (4.2) is *strictly weakly* (resp., *strongly*) *dissipative with respect to the* (locally Lebesgue integrable) *supply rate* $s : U \times Y \to \mathbb{R}$ if there exist a locally Lipschitz continuous, regular, and nonnegative storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}$ and a scalar $\varepsilon > 0$ such that $V_{\rm s}(0) = 0$ and the dissipation inequality

$$V_{\rm s}(x(t)) \leq V_{\rm s}(x(t_0)) + \int_{t_0}^{t} [s(u(\sigma),y(\sigma)) - \varepsilon]{\rm d}\sigma, \quad t \geq t_0, \tag{4.5}$$

is satisfied for at least one (resp., every) Filippov solution $x(t)$, $t \geq t_0$, of $\mathcal{G}$ with $u(t) \in U$.

Since $V_{\rm s}(\cdot)$ is assumed to be locally Lipschitz continuous and regular, an equivalent statement for the dissipativeness of $\mathcal{G}$ involving supply rates $s(u,y)$ is

$$\dot{V}_{\rm s}(x(t)) \leq s(u(t),y(t)), \text{ a.e.} \quad t \geq 0, \tag{4.6}$$

or, equivalently, $\dot{V}_{\rm s}(x) \leq s(u,y)$, where

$$\dot{V}_{\rm s}(x) = \frac{\rm d}{{\rm d}t}V_{\rm s}(\psi(t,x,u))\bigg|_{t=0} \triangleq \limsup_{h\to 0^+} \frac{V_{\rm s}(\psi(h,x,u)) - V_{\rm s}(x)}{h}, \tag{4.7}$$

for every $x \in \mathbb{R}^n$, denotes the upper right directional Dini derivative of $V_{\rm s}(x)$ along the Filippov state trajectories $\psi(t,x,u)$ of (4.1) through $x \in \mathcal{D}$ with $u(t) \in U$ at $t = 0$. Alternatively, an equivalent statement for exponentional dissipativeness and strict dissipativeness of $\mathcal{G}$ involving the supply rate $s(u,y)$ is, respectively,

$$\dot{V}_{\rm s}(x(t)) + \varepsilon V_{\rm s}(x(t)) \leq s(u(t),y(t)), \text{ a.e.} \quad t \geq 0, \tag{4.8}$$

and

$$\dot{V}_{\rm s}(x(t)) \leq s(u(t), y(t)) - \varepsilon, \text{ a.e.} \quad t \geq 0. \tag{4.9}$$

The following lemma is necessary for the next proposition. For the statement of this lemma we require some additional notation. Specifically, given a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, define the set-valued Lie derivative $\mathcal{L}_{F(x,u)}V : \mathbb{R}^n \times U \to 2^{\mathbb{R}}$ of V with respect to F at x and u by

$$\mathcal{L}_{F(\cdot,u)}V(x) \triangleq \Big\{a \in \mathbb{R} : \text{there exists } v \in \mathcal{K}[F(\cdot,u)](x) \\ \text{such that } p^{\rm T}v = a \text{ for all } p^{\rm T} \in \partial V(x)\Big\},$$

where $\mathcal{K}[F(\cdot,u)](x)$ denotes the Filippov set-valued map of $F(x,u)$ over x for each admissible input $u(t) \in U$. That is, $F(\cdot,u)$ is averaged over progressively smaller neighborhoods around $x \in \mathbb{R}^n$ with $u \in U$. Analogously, for fixed $t > 0$, $x \in \mathbb{R}^n$, and a measurable and locally essentially bounded $u : \mathbb{R} \to U$, define the set-valued Lie derivative $\mathcal{L}_{F(x,u(\cdot))}V : \mathbb{R}^n \times U \to 2^{\mathbb{R}}$ by

$$\mathcal{L}_{F(\cdot,u(\cdot))}V(x) \triangleq \Big\{a \in \mathbb{R} : \text{there exists } v \in \mathcal{K}[F(\cdot,u(t))](x) \\ \text{such that } p^{\rm T}v = a \text{ for all } p^{\rm T} \in \partial V(x)\Big\},$$

that is, we fix $u(\cdot) \in \mathcal{L}_\infty(\mathbb{R},U)$ and apply the Filippov construction over x. Note that if $\psi(\cdot)$ is a Filippov solution to (4.1) with $u(\cdot) = \overline{u}(\cdot)$, then $\mathcal{L}_{F(\cdot,\overline{u}(\cdot))}V(\psi(t)) \subseteq \mathcal{L}_{F(\cdot,u)}V(\psi(t))$. In addition, note that $\mathcal{L}_{F(\cdot,u)}V(x)$ is a closed and bounded, possibly empty, interval in $\mathbb{R}$.

Lemma 4.2.1. Let $x : [t_0,t] \to \mathbb{R}^n$ be a Filippov solution of (4.1) corresponding to the input $u(\cdot)$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous and regular. Then $\frac{\rm d}{{\rm d}\sigma}V(x(\sigma))$ exists for almost all $\sigma \in [t_0,t]$ and $\frac{\rm d}{{\rm d}\sigma}V(x(\sigma)) \in \mathcal{L}_{F(\cdot,u(\cdot))}V(x(\sigma))$ for almost all $\sigma \in [t_0,t]$.

Proof. The proof is similar to the proof of Lemma 1 of [5] and, hence, is omitted. □

Proposition 4.2.1. Consider the discontinuous dynamical system $\mathcal{G}$ given by (4.1) and (4.2), and let $V : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz continuous and regular function such that $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$. Assume there exist a Lebesgue measurable function $s : U \times Y \to \mathbb{R}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that

$$\max \mathcal{L}_{F(\cdot,u)} V(x) \leq -\varepsilon V(x) + s(u, y), \quad \text{a.e.} \quad u \in U. \tag{4.10}$$

Then $\mathcal{G}$ is strongly exponentially dissipative (resp., strongly dissipative) with respect to the supply rate $s(u, y)$.

Proof. It suffices to show that if (4.10) holds, then (4.3) holds on the interval $[t_0, t]$. To see this, let $x : [t_0, t] \to \mathbb{R}^n$ be a Filippov solution of (3.3) with initial condition $x(0) = x_0$. Now, since by Lemma 4.2.1 $\dot{V}(x(\sigma)) \leq \max \mathcal{L}_{\mathcal{F}(\cdot,u(\cdot))} V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$, it follows from (4.10) that $\dot{V}(x(\sigma)) \leq -\varepsilon V(x(\sigma)) + s(u(\sigma), y(\sigma))$ for almost all $\sigma \in [t_0, t]$, and hence,

$$e^{\varepsilon\sigma} \left[\dot{V}(x(\sigma)) + \varepsilon V(x(\sigma)) \right] \leq e^{\varepsilon\sigma} s(u(\sigma), y(\sigma)), \quad \text{a.e.} \quad \sigma \in [t_0, t]. \tag{4.11}$$

Now, integrating (4.11), where the integral is a Lebesgue integral, it follows that (4.3) holds with $\varepsilon > 0$ (resp., $\varepsilon = 0$). □

Example 4.2.1. Consider the controlled discontinuous dynamical system $\mathcal{G}$ representing a mass sliding on a horizontal surface subject to a Coulomb frictional force. During sliding, the Coulomb frictional model states that the magnitude of the friction force is independent of the magnitude of the system velocity and is equal to the normal contact force times the coefficient of kinetic friction. The application of this model to a sliding mass on a horizontal frictional surface gives

$$\dot{x}(t) = -\mathrm{sign}(x(t)) + u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.12}$$

$$y(t) = x(t). \tag{4.13}$$

Equation (4.12) can be rewritten in the form of a differential inclusion

$$\dot{x}(t) \in \mathcal{K}[f](x(t)) + u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.14}$$

where the Filippov set-valued map $\mathcal{K}[f] : \mathbb{R} \to 2^{\mathbb{R}}$ is given by

$$\mathcal{K}[f](x) = \begin{cases} -1, & x > 0, \\ [-1, 1], & x = 0, \\ 1, & x < 0. \end{cases} \tag{4.15}$$

Let $V_{\mathrm{s}_1}(x) = x^2$. Since

$$\begin{aligned} \dot{V}_{\mathrm{s}_1}(x) \in\ & \mathcal{L}_{F(\cdot,u)} V_{s1}(x) \\ =\ & \partial V_{\mathrm{s}_1}(x)(\mathcal{K}[f](x) + u) \\ =\ & 2x\mathcal{K}[f](x) + 2xu \\ =\ & -|x| + 2uy \\ \leq\ & 2uy, \end{aligned} \tag{4.16}$$

it follows that $\max \mathcal{L}_{F(\cdot,u)} V_{\mathrm{s}_1}(x) \leq 2uy$ for all Filippov solutions, which, by Proposition 4.2.1, implies that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $2uy$.

Next, let $V_{\mathrm{s}_2}(x) = |x|$. Since

$$\begin{aligned} \dot{V}_{\mathrm{s}_2}(x) \in \mathcal{L}_{F(\cdot,u)} V_{\mathrm{s}_2}(x) &= \begin{cases} -1 + \mathrm{sign}(x)u, & x \neq 0, \\ 0, & x = 0, \end{cases} \\ &= -1 + u\,\mathrm{sign}(y), \quad x \neq 0, \end{aligned} \tag{4.17}$$

it follows that $\max \mathcal{L}_{F(\cdot,u)} V_{\mathrm{s}_2}(x) \leq u\,\mathrm{sign}(y)$ for almost all $x \in \mathbb{R}$ and all Filippov solutions, which, by Proposition 4.2.1, implies that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $u\,\mathrm{sign}(y)$. △

4.3. Extended Kalman–Yakubovich–Popov conditions

Next, we show that dissipativeness of discontinuous nonlinear affine dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{4.18}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{4.19}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $h : \mathcal{D} \to Y$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$, can be characterized in terms of the system functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$. Here, we assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are Lebesgue measurable and locally essentially bounded.

For the remainder of this section, we consider the special case of dissipative systems with quadratic supply rates [75], [31]. Specifically, set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^l$, let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l\times m}$ be given, and assume $s(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$, where $\mathbb{S}^q$ denotes the set of $q \times q$ symmetric matrices. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $s(\kappa(y), y) < 0$, $y \neq 0$, so that, as shown by Theorem 3.2 of [38], all storage functions for $\mathcal{G}$ are positive definite. Next, define

$$\mathcal{L}_G V_{\mathrm{s}}(x) \triangleq \{q \in \mathbb{R}^{1\times m} : \text{there exists } v \in \mathfrak{G}(x) \text{ such that } p^{\mathrm{T}}v = q \text{ for all } p^{\mathrm{T}} \in \partial V_{\mathrm{s}}(x)\},$$

where $\mathfrak{G}(x) \triangleq \bigcap_{\delta>0}\bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\{G(\mathcal{B}_\delta(x))\backslash\mathcal{S}\}$, $x \in \mathbb{R}^n$, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. Finally, we assume that the set $\mathcal{L}_G V_{\mathrm{s}}(x)$ is single-valued[4] for almost all $x \in \mathbb{R}^n$ modulo $\mathcal{L}_G V_{\mathrm{s}}(x) \neq \varnothing$. The following definition is necessary for the statement of the next result.

[4] The assumption that $\mathcal{L}_G V_{\mathrm{s}}(x)$ is single-valued is necessary for obtaining Kalman-Yakubovich-Popov conditions for (4.18) and (4.19) with Lebesgue measurable and locally essentially bounded system functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$, and with locally Lipschitz continuous storage functions $V_{\mathrm{s}}(\cdot)$. Specifically, as will be seen in the proof of Theorem 4.3.1, the requirement that there ex-

Definition 4.3.1. ([38]) The nonlinear dynamical system $\mathcal{G}$ given by (4.1) and (4.2) is *weakly (*resp., *strongly) completely reachable* if for every $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ there exists a finite time $t_{\rm i} < t_0$ and an admissible input $u(t)$ defined on $[t_{\rm i}, t_0]$ such that at least one (resp., every) Filippov solution $x(t)$, $t \geq t_{\rm i}$, of $\mathcal{G}$ can be driven from $x(t_{\rm i}) = 0$ to $x(t_0) = x_0$. The nonlinear dynamical system $\mathcal{G}$ given by (2.1) and (2.2) is *weakly (*resp., *strongly) zero-state observable* if $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $x(t) \equiv 0$ for at least one (resp., every) Filippov solution of $\mathcal{G}$.

The following theorem gives necessary and sufficient Kalman–Yakubovich–Popov conditions for dynamical systems with Lebesgue measurable and locally essentially bounded system functions.

Theorem 4.3.1. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,

$$0 = \min \mathcal{L}_f V_{\rm s}(x) + \varepsilon V_{\rm s}(x) - h^{\rm T}(x)Qh(x) + \ell^{\rm T}(x)\ell(x), \tag{4.20}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\rm s}(x) - h^{\rm T}(x)(QJ(x) + S) + \ell^{\rm T}(x)\mathcal{W}(x), \tag{4.21}$$

$$0 = R + S^{\rm T}J(x) + J^{\rm T}(x)S + J^{\rm T}(x)QJ(x) - \mathcal{W}^{\rm T}(x)\mathcal{W}(x), \tag{4.22}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\rm T}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\rm s}(x) - \min \mathcal{L}_f V_{\rm s}(x), \quad u \in \mathbb{R}^m, \tag{4.23}$$

then $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate $s(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$. Conversely, if $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate

ists $\overline{z} \in \mathcal{L}_G V_{\rm s}(x)$ (resp., $\underline{z} \in \mathcal{L}_G V_{\rm s}(x)$) such that, for all $u \in \mathbb{R}^m$, $\max[\mathcal{L}_G V_{\rm s}(x)u] = \overline{z}u$ (resp., $\min[\mathcal{L}_G V_{\rm s}(x)u] = \underline{z}u$) used in the proof of Theorem 4.3.1 holds if and only if $\mathcal{L}_G V_{\rm s}(x)$ is a singleton. This fact is shown in Footnote 3 for $\overline{z} \in \mathcal{L}_G V_{\rm s}(x)$. A similar construction shows the result for $\underline{z} \in \mathcal{L}_G V_{\rm s}(x)$.

$s(u,y)$, then there exist functions $V_{\rm s}:\mathbb{R}^n\to\mathbb{R}$, $\ell:\mathbb{R}^n\to\mathbb{R}^p$, and $\mathcal{W}:\mathbb{R}^n\to\mathbb{R}^{p\times m}$ and a scalar $\varepsilon>0$ (resp., $\varepsilon=0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0)=0$, and, for almost all $x\in\mathbb{R}^n$, (4.20)–(4.22) hold.

Proof. First, suppose that there exist functions $V_{\rm s}:\mathbb{R}^n\to\mathbb{R}$, $\ell:\mathbb{R}^n\to\mathbb{R}^p$, and $\mathcal{W}:\mathbb{R}^n\to\mathbb{R}^{p\times m}$ and a scalar $\varepsilon>0$ such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, and (4.20)–(4.23) are satisfied. Then, for every admissible input $u(t)\in\mathbb{R}^m$, $t\geq 0$, it follows from (4.20)–(4.23) that

$$\begin{aligned}
&\int_{t_1}^{t_2} e^{\varepsilon t}s(u(t),y(t)){\rm d}t\\
&\quad=\int_{t_1}^{t_2} e^{\varepsilon t}\big[y^{\rm T}(t)Qy(t)+2y^{\rm T}(t)Su(t)+u^{\rm T}(t)Ru(t)\big]{\rm d}t\\
&\quad=\int_{t_1}^{t_2} e^{\varepsilon t}\big[h^{\rm T}(x(t))Qh(x(t))+2h^{\rm T}(x(t))(S+QJ(x(t)))u(t)\\
&\qquad+u^{\rm T}(t)(J^{\rm T}(x(t))QJ(x(t))+S^{\rm T}J(x(t))+J^{\rm T}(x(t))S+R)u(t)\big]{\rm d}t\\
&\quad=\int_{t_1}^{t_2} e^{\varepsilon t}\big[\min\mathcal{L}_fV_{\rm s}(x(t))+\varepsilon V_{\rm s}(x(t))+\mathcal{L}_GV_{\rm s}(x(t))u(t)+\ell^{\rm T}(x(t))\ell(x(t))\\
&\qquad+2\ell^{\rm T}(x(t))\mathcal{W}(x(t))u(t)+u^{\rm T}(t)\mathcal{W}^{\rm T}(x(t))\mathcal{W}(x(t))u(t)\big]{\rm d}t\\
&\quad=\int_{t_1}^{t_2} e^{\varepsilon t}[\min\mathcal{L}_fV_{\rm s}(x(t))+\mathcal{L}_GV_{\rm s}(x(t))u(t)+\varepsilon V_{\rm s}(x(t))\\
&\qquad+[\ell(x(t))+\mathcal{W}(x(t))u(t)]^{\rm T}[\ell(x(t))+\mathcal{W}(x(t))u(t)]\big]{\rm d}t\\
&\quad\geq\int_{t_1}^{t_2} e^{\varepsilon t}[\max\mathcal{L}_fV_{\rm s}(x(t))+\mathcal{L}_GV_{\rm s}(x(t))u(t)+\varepsilon V_{\rm s}(x(t))]{\rm d}t,
\end{aligned}\tag{4.24}$$

where $x(t)$, $t\geq 0$, satisfies (4.18).

Next, using the sum rule for computing the generalized gradient of a locally Lipschitz continuous function [56] it follows that

$$\mathcal{L}_{f+Gu}V_{\rm s}(x)\subseteq\mathcal{L}_fV_{\rm s}(x)+\mathcal{L}_{Gu}V_{\rm s}(x)$$

for almost all $x\in\mathbb{R}^n$. Now, it follows from Lemma 4.2.1 that

$$\frac{{\rm d}}{{\rm d}t}V_{\rm s}(x(t))\in\mathcal{L}_{f+Gu}V_{\rm s}(x(t))\subseteq\mathcal{L}_fV_{\rm s}(x(t))+\mathcal{L}_{Gu}V_{\rm s}(x(t))$$

for almost all $t \geq 0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) &\leq \max \mathcal{L}_{f+Gu}V_{\mathrm{s}}(x(t)) \\ &\leq \max\left[\mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t)\right] \\ &= \max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t), \quad \text{a.e. } t \geq 0, \ \ u(t) \in U. \quad (4.25)\end{aligned}$$

Next, note that

$$e^{\varepsilon t}V_{\mathrm{s}}(x(t)) = e^{\varepsilon t_0}V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma}(e^{\varepsilon\sigma}V_{\mathrm{s}}(x(\sigma)))\mathrm{d}\sigma, \tag{4.26}$$

where the integral in (4.26) is the Lebesgue integral.

Using (4.25) and (4.26), it follows from (4.24) that

$$\begin{aligned}\int_{t_1}^{t_2} e^{\varepsilon t}s(u(t),y(t))\mathrm{d}t &\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\ &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(e^{\varepsilon t}V_{\mathrm{s}}(x(t)))\mathrm{d}t \\ &= e^{\varepsilon t_2}V_{\mathrm{s}}(x(t_2)) - e^{\varepsilon t_1}V_{\mathrm{s}}(x(t_1)), \quad \text{a.e.} \quad t \geq 0, \quad u(t) \in U.\end{aligned}$$

The assertion now follows from Definition 4.2.1.

Conversely, suppose that $\mathcal{G}$ is weakly exponentially dissipative with respect to the supply rate $s(u,y)$. Now, it follows from Theorem 3.1 of [38] that the available storage $V_{\mathrm{as}}(x)$ of $\mathcal{G}$ is finite for all $x \in \mathbb{R}^n$, $V_{\mathrm{as}}(0) = 0$, and

$$e^{\varepsilon t_2}V_{\mathrm{as}}(x(t_2)) \leq e^{\varepsilon t_1}V_{\mathrm{as}}(x(t_1)) + \int_{t_1}^{t_2} e^{\varepsilon t}s(u(t),y(t))\mathrm{d}t \tag{4.27}$$

for almost all $t_2 \geq t_1$ and $u(\cdot) \in \mathcal{U}$. Dividing (4.27) by $t_2 - t_1$ and letting $t_2 \to t_1$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x(t)) + \varepsilon V_{\mathrm{as}}(x(t)) \leq s(u(t),y(t)), \ \text{a.e. } t \geq 0, \tag{4.28}$$

where $x(t)$, $t \geq 0$, is a solution satisfying (4.18) and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x(t)) = \limsup_{h\to 0^+}[V_{\mathrm{as}}(x(t+h)) - V_{\mathrm{as}}(x(t))]/h.$$

Now, with $t = 0$, it follows from (4.28) that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x_0) + \varepsilon V_{\mathrm{as}}(x_0) \le s(u, y(0)), \qquad u \in \mathbb{R}^m.$$

Next, let $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be such that

$$d(x, u) \triangleq -\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) - \varepsilon V_{\mathrm{as}}(x) + s(u, y). \tag{4.29}$$

Now, it follows from (4.28) that $d(x, u) \ge 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Since $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) \in \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_{Gu} V_{\mathrm{as}}(x)$ for almost all $x \in \mathbb{R}^n$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) \ge \min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u, \quad \text{a.e.} \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{4.30}$$

and hence, it follows from (4.29) and (4.30) that

$$-[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + s(u, h(x) + J(x)u) \ge d(x, u) \ge 0, \quad \text{a.e. } x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{4.31}$$

Since the left-hand side of (4.31) is quadratic in u, there exist functions $\ell : \mathbb{R}^n \to \mathbb{R}^p$ and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that

$$\begin{aligned}
\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \\
&= -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + s(u, h(x) + J(x)u) \\
&= -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + [h(x) + J(x)u]^{\mathrm{T}} \\
&\quad \cdot Q[h(x) + J(x)u] + 2[h(x) + J(x)u]^{\mathrm{T}} S u + u^{\mathrm{T}} R u.
\end{aligned}$$

Now, equating coefficients of equal powers yields (4.20)–(4.22) with $V_{\mathrm{s}}(x) = V_{\mathrm{as}}(x)$ and with the positive definiteness of $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, following from Theorem 3.2 of [38].

Finally, the proof for the weakly dissipative case follows by using an identical construction with $\varepsilon = 0$. □

Remark 4.3.1. Note that if $\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$ is invertible for all $x \in \mathbb{R}^n$, then inequality (4.23) can be equivalently written as

$$[\ell(x) - \mathcal{W}(x)(\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x))^{-1}\mathcal{W}^{\mathrm{T}}(x)\ell(x)]^{\mathrm{T}}[\ell(x) - \mathcal{W}(x)(\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x))^{-1}\mathcal{W}^{\mathrm{T}}(x)\ell(x)] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad x \in \mathbb{R}^n, \qquad (4.32)$$

which is free of $u \in \mathbb{R}^m$. This follows from the fact that (4.23) holds if and only if

$$\min_u [\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad x \in \mathbb{R}^n, \quad (4.33)$$

holds. A similar expression to (4.32) involving generalized inverses also holds in the case where $\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$ is singular for some $x \in \mathbb{R}^n$.

The following result gives sufficient conditions for weak dissipativity and weak exponential dissipativity of $\mathcal{G}$ based on $\max \mathcal{L}_f V_{\mathrm{s}}(\cdot)$.

Theorem 4.3.2. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,

$$0 = \max \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \qquad (4.34)$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \qquad (4.35)$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \qquad (4.36)$$

then $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate $s(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$.

Proof. Suppose that there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, and (4.34)–(4.36) are satisfied. Then, for every admissible input $u(t) \in \mathbb{R}^m$, it follows from (4.34)–(4.36) and (4.25) that

$$\begin{aligned}
&\int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t))\mathrm{d}t \\
&\qquad =\int_{t_1}^{t_2} e^{\varepsilon t}\big[y^{\mathrm{T}}(t)Qy(t) + 2y^{\mathrm{T}}(t)Su(t) + u^{\mathrm{T}}(t)Ru(t)\big]\mathrm{d}t \\
&\qquad =\int_{t_1}^{t_2} e^{\varepsilon t}\big[h^{\mathrm{T}}(x(t))Qh(x(t)) + 2h^{\mathrm{T}}(x(t))(S + QJ(x(t)))u(t) \\
&\qquad\quad +u^{\mathrm{T}}(t)(J^{\mathrm{T}}(x(t))QJ(x(t)) + S^{\mathrm{T}}J(x(t)) + J^{\mathrm{T}}(x(t))S + R)u(t)\big]\mathrm{d}t \\
&\qquad =\int_{t_1}^{t_2} e^{\varepsilon t}[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t)) \\
&\qquad\quad +[\ell(x(t)) + \mathcal{W}(x(t))u(t)]^{\mathrm{T}}[\ell(x(t)) + \mathcal{W}(x(t))u(t)]\big]\mathrm{d}t \\
&\qquad \geq\int_{t_1}^{t_2} e^{\varepsilon t}[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t))]\mathrm{d}t \\
&\qquad \geq\int_{t_1}^{t_2} e^{\varepsilon t}\left[\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\
&\qquad =e^{\varepsilon t_2}V_{\mathrm{s}}(x(t_2)) - e^{\varepsilon t_1}V_{\mathrm{s}}(x(t_1)), \quad \text{a.e.} \quad t \geq 0,
\end{aligned}$$

where $x(t)$, $t \geq t_0$, is a solution satisfying (4.18). The result is now immediate from Definition 4.2.1. The proof for the weak dissipative case follows an identical construction by setting $\varepsilon = 0$. □

Next, we provide several definitions of nonlinear discontinuous dynamical systems which are dissipative or exponentially dissipative with respect to supply rates of a specific form.

Definition 4.3.2. A discontinuous dynamical system $\mathcal{G}$ of the form (4.1) and (4.2) with $m = l$ is *weakly* (resp., *strongly*) *passive* if $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the supply rate $s(u, y) = 2u^{\mathrm{T}}y$.

Definition 4.3.3. A discontinuous dynamical system $\mathcal{G}$ of the form (4.1) and (4.2) is *weakly* (resp., *strongly*) *nonexpansive* if $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the supply rate $s(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

Definition 4.3.4. A discontinuous dynamical system $\mathcal{G}$ of the form (4.1) and (4.2) with $m = l$ is *weakly* (resp., *strongly*) *exponentially passive* if $\mathcal{G}$ is weakly (resp., strongly) exponentially dissipative with respect to the supply rate $s(u, y) = 2u^{\mathrm{T}}y$.

Definition 4.3.5. A discontinuous dynamical system $\mathcal{G}$ of the form (4.1) and (4.2) is *weakly* (resp., *strongly*) *exponentially nonexpansive* if $\mathcal{G}$ is weakly (resp., strongly) exponentially dissipative with respect to the supply rate $s(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

The following results present the nonlinear versions of the Kalman-Yakubovich-Popov *strict positive real lemma* (resp., *positive real lemma*) and *strict bounded real lemma* (resp., *bounded real lemma*) for weakly exponentially passive (resp., weakly passive) and weakly exponentially nonexpansive (resp., weakly nonexpansive) discontinuous systems, respectively.

Corollary 4.3.1. Let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,

$$0 = \min \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{4.37}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{4.38}$$

$$0 = J(x) + J^{\mathrm{T}}(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{4.39}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad u \in \mathbb{R}^m, \tag{4.40}$$

then $\mathcal{G}$ is weakly exponentially passive (resp., weakly passive). Conversely, if $\mathcal{G}$ is weakly exponentially passive (resp., weakly passive), then there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (4.37)–(4.39) hold.

Proof. The result is a direct consequence of Theorem 4.3.1 with $l = m$, $Q = 0$, $S = I_m$, and $R = 0$. Specifically, with $\kappa(y) = -y$ it follows that $s(\kappa(y), y) = -2y^{\rm T}y < 0$, $y \neq 0$, so that all the assumptions of Theorem 4.3.1 are satisfied. □

Example 4.3.1. Consider the harmonic oscillator $\mathcal{G}$ with Coulomb friction given by ([67])

$$m\ddot{x}(t) + b\,\mathrm{sign}(\dot{x}(t)) + kx(t) = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad \text{a.e. } t \geq 0, \tag{4.41}$$

$$y(t) = \frac{1}{2}\dot{x}(t), \tag{4.42}$$

or, equivalently,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_1(t) - \frac{b}{m}\,\mathrm{sign}(x_2(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad \text{a.e.} \quad t \geq 0, \tag{4.43}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{4.44}$$

where $m, b, k > 0$. Next, consider the continuously differentiable storage function $V_{\rm s}(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$ and note that, for almost all $x \in \mathbb{R}^2$, $\mathcal{L}_f V_{\rm s}(x) = \{-b|x_2|\}$ and $\mathcal{L}_G V_{\rm s}(x) = \{x_2\}$, which implies that $\min \mathcal{L}_f V_{\rm s}(x) = \max \mathcal{L}_f V_{\rm s}(x) = -b|x_2|$. Now, with $\ell(x) = \pm\sqrt{b|x_2|}$ and $\mathcal{W}(x) = 0$, (4.37)–(4.40) are satisfied. Hence, it follows from Corollary 4.3.1 that $\mathcal{G}$ is weakly passive. △

Example 4.3.2. Consider a controlled smooth oscillator with nonsmooth friction and uncertain coefficients given in [5] represented by the differential inclusion $\mathcal{G}$ given by

$$\dot{x}(t) \in \mathcal{K}[f](x(t)) + Gu(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.45}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{4.46}$$

where $G = [0,1]^{\mathrm{T}}$ and $\mathcal{K}[f] : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ is given by

$$\mathcal{K}[f](x) \triangleq \begin{cases} [-2x_2 - 1, -x_2 - 1] \times \{x_1\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \\ \{-x_2 - \operatorname{sign}(x_1)\} \times \{x_1\}, & (x_1, x_2) \in \mathbb{R}^2 \backslash (\{(0, x_2) : x_2 \in \mathbb{R}\} \\ & \quad \cup \{(x_1, x_2) : x_1 > 0, x_2 > 0\}), \\ [-2x_2 - 1, -x_2 + 1] \times \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0, x_1 = 0, \\ [-x_2 - 1, -x_2 + 1] \times \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 < 0, x_1 = 0, \\ [-1, 1] \times \{0\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Next, consider the continuously differentiable storage function $V_{\mathrm{s}}(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and note that for almost all $x \in \mathbb{R}^2$,

$$\mathcal{L}_f V_{\mathrm{s}}(x) = \begin{cases} \{[-1, 0]x_1 x_2 - x_1\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \\ \{-|x_1|\}, & (x_1, x_2) \in \mathbb{R}^2 \backslash (\{(0, x_2) : x_2 \in \mathbb{R}\} \\ & \quad \cup \{(x_1, x_2) : x_1 > 0, x_2 > 0\}), \\ \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

$$\mathcal{L}_G V_{\mathrm{s}}(x) = \{x_2\},$$

which implies that $\max \mathcal{L}_f V_{\mathrm{s}}(x) = 0$ and $\min \mathcal{L}_f V_{\mathrm{s}}(x) = -|x_1|$ for almost all $x \in \mathbb{R}^2$. Now, it follows from (4.37)–(4.40) that

$$0 = -|x_1| + \ell^2(x), \tag{4.47}$$

$$0 = \frac{1}{2}x_2 - \frac{1}{2}x_2 + \ell(x)\mathcal{W}(x), \tag{4.48}$$

$$0 = \mathcal{W}^2(x), \tag{4.49}$$

$$|x_1| \leq [\ell(x) + \mathcal{W}(x)u]^2, \quad u \in \mathbb{R}. \tag{4.50}$$

Hence, with $\ell(x) = \pm\sqrt{|x_1|}$ and $\mathcal{W}(x) = 0$, it follows from Corollary 4.3.1 that $\mathcal{G}$ is weakly passive. △

Example 4.3.3. Consider a controlled nonsmooth harmonic oscillator with nonsmooth friction and nonsmooth output given by ([5])

$$\dot{x}(t) = f(x(t)) + Gu(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.51}$$

$$y(t) = \frac{1}{2}\operatorname{sign}(x_2(t)), \tag{4.52}$$

where $f(x) = [-\operatorname{sign}(x_2) - \frac{1}{2}\operatorname{sign}(x_1), \operatorname{sign}(x_1)]^{\mathrm{T}}$ and $G = [0, 1]^{\mathrm{T}}$. Next, consider the locally Lipschitz continuous storage function $V_{\mathrm{s}}(x) = |x_1| + |x_2|$ and note that

$$\partial V_{\mathrm{s}}(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \overline{\mathrm{co}}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V_{\mathrm{s}}(x_1, x_2) = \begin{cases} \{-\frac{1}{2}\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

$$\mathcal{L}_G V_{\mathrm{s}}(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

which implies that $\max \mathcal{L}_f V_{\mathrm{s}}(x) = 0$, $\min \mathcal{L}_f V_{\mathrm{s}}(x) = -\frac{1}{2}$, and $\mathcal{L}_G V_{\mathrm{s}}(x) = \{\operatorname{sign}(x_2)\}$ for almost all $x \in \mathbb{R}^2$. Now, it follows from (4.37)–(4.40) that

$$0 = -\frac{1}{2} + \ell^2(x), \tag{4.53}$$

$$0 = \frac{1}{2}\operatorname{sign}(x_2) - \frac{1}{2}\operatorname{sign}(x_2) + \ell(x)\mathcal{W}(x), \tag{4.54}$$

$$0 = \mathcal{W}^2(x), \tag{4.55}$$

$$\frac{1}{2} \leq [\ell(x) + \mathcal{W}(x)u]^2, \quad u \in \mathbb{R}. \tag{4.56}$$

Hence, with $\ell(x) = \pm\sqrt{\frac{1}{2}}$ and $\mathcal{W}(x) = 0$, it follows from Corollary 4.3.1 that $\mathcal{G}$ is weakly passive. △

Corollary 4.3.2. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l \times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$,

$\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,

$$0 = \min \mathcal{L}_f V_{\rm s}(x) + \varepsilon V_{\rm s}(x) + h^{\rm T}(x)h(x) + \ell^{\rm T}(x)\ell(x), \tag{4.57}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\rm s}(x) + h^{\rm T}(x)J(x) + \ell^{\rm T}(x)\mathcal{W}(x), \tag{4.58}$$

$$0 = \gamma^2 I_m - J^{\rm T}(x)J(x) - \mathcal{W}^{\rm T}(x)\mathcal{W}(x), \tag{4.59}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\rm T}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\rm s}(x) - \min \mathcal{L}_f V_{\rm s}(x), \quad u \in \mathbb{R}^m, \tag{4.60}$$

where $\gamma > 0$, then $\mathcal{G}$ is weakly exponentially nonexpansive (resp., weakly nonexpansive). Conversely, if $\mathcal{G}$ is weakly exponentially nonexpansive (resp., weakly nonexpansive), then there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (4.57)–(4.59) hold.

Proof. The result is a direct consequence of Theorem 4.3.1 with $Q = -I_l$, $S = 0$, and $R = \gamma^2 I_m$. Specifically, with $\kappa(y) = -\frac{1}{2\gamma}y$ it follows that $s(\kappa(y), y) = -\frac{3}{4}y^{\rm T}y < 0$, $y \neq 0$, so that all the assumptions of Theorem 4.3.1 are satisfied. □

Example 4.3.4. Consider the controlled dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.61}$$

$$y(t) = x(t), \tag{4.62}$$

where $x(t) = [x_1(t), x_2(t)]^{\rm T}$, $u(t) = [u_1(t), u_2(t)]^{\rm T}$,

$$f(x) = \begin{bmatrix} |x_1|(-x_1 + |x_2|) \\ x_2(-x_1 - |x_2|) \end{bmatrix}, \quad G(x) = \begin{bmatrix} |x_1| & 0 \\ 0 & x_2 \end{bmatrix}.$$

Next, consider the locally Lipschitz continuous storage function $V_{\rm s}(x) = 2|x_1| + 2|x_2|$

and note that

$$\partial V_{\rm s}(x_1,x_2) = \begin{cases} \{2\,\mathrm{sign}(x_1)\}\times\{2\,\mathrm{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2\neq 0,\\ \{2\,\mathrm{sign}(x_1)\}\times[-2,2], & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2 = 0,\\ [-2,2]\times\{2\,\mathrm{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2\neq 0, x_1 = 0,\\ \overline{\mathrm{co}}\{(2,2),(-2,2),(-2,-2),(2,-2)\}, & (x_1,x_2)=(0,0). \end{cases}$$

Hence,

$$\mathcal{L}_f V_{\rm s}(x_1,x_2) = \begin{cases} \{-2x_1^2-2x_2^2\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2\neq 0,\\ \{-2x_1^2\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2 = 0,\\ \{-2x_2^2\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2\neq 0, x_1 = 0,\\ \{0\}, & (x_1,x_2)=(0,0), \end{cases}$$

$$\mathcal{L}_G V_{\rm s}(x_1,x_2) = \begin{cases} \{[2x_1, 2|x_2|]\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2\neq 0,\\ \{[2x_1, 0]\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2 = 0,\\ \{[0, 2|x_2|]\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2\neq 0, x_1 = 0,\\ \{[0,0]\}, & (x_1,x_2)=(0,0), \end{cases}$$

which implies that

$$\min \mathcal{L}_f V_{\rm s}(x) = \max \mathcal{L}_f V_{\rm s}(x) = -2x_1^2 - 2x_2^2 \ \text{ and } \ \mathcal{L}_G V_{\rm s}(x) = \{[2x_1, 2|x_2|]\}$$

for almost all $x\in\mathbb{R}^2$. Now, it follows from (4.57)–(4.60) that

$$0 = -2x_1^2 - 2x_2^2 + x_1^2 + x_2^2 + \ell^{\rm T}(x)\ell(x), \tag{4.63}$$

$$0 = \frac{1}{2}[2x_1, 2|x_2|] + \ell^{\rm T}(x)\mathcal{W}(x), \tag{4.64}$$

$$0 = \gamma^2 I_2 - \mathcal{W}^{\rm T}(x)\mathcal{W}(x), \tag{4.65}$$

$$0 \leq [\ell(x)+\mathcal{W}(x)u]^{\rm T}[\ell(x)+\mathcal{W}(x)u], \quad u\in\mathbb{R}^2. \tag{4.66}$$

Hence, with $\gamma = 1$, $\ell(x) = -[x_1, |x_2|]^{\rm T}$, and $\mathcal{W}(x) = I_2$, it follows from Corollary 4.3.2 that $\mathcal{G}$ is weakly nonexpansive. △

In light of Definition 4.3.2 the following result is immediate.

Proposition 4.3.1. Consider the discontinuous dynamical system $\mathcal{G}$ given by (4.1) and (4.2). Then the following statements hold:

i) If $\mathcal{G}$ is strongly passive with a locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly Lyapunov stable.

ii) If $\mathcal{G}$ is strongly exponentially passive with a locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.

iii) If $\mathcal{G}$ is strongly zero-state observable and strongly nonexpansive with locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.

iv) If $\mathcal{G}$ is strongly exponentially nonexpansive with a locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.

Proof. Statements *i*)–*iv*) are immediate and follow from (4.6)–(4.8) using Lyapunov and invariant set stability arguments given by Theorems 2.2.1 and 2.2.2, respectively. □

4.4. Stability of Feedback Interconnections of Dissipative Discontinuous Dynamical Systems

In this section, we consider feedback interconnections of dissipative discontinuous dynamical systems. Specifically, using the notions of dissipativity and exponential dissipativity for discontinuous dynamical systems, with appropriate storage functions and supply rates, we construct (not necessarily smooth) Lyapunov functions for interconnected discontinuous dynamical systems by appropriately combining the storage functions for each subsystem.

We begin by considering the nonlinear discontinuous dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.67}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{4.68}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^l$, and $J : \mathbb{R}^n \to \mathbb{R}^{l\times m}$, with the nonlinear feedback discontinuous system $\mathcal{G}_c$ given by

$$\dot{x}_c(t) = f_c(x_c(t)) + G_c(u_c(t), x_c(t))u_c(t), \quad x_c(0) = x_{c0}, \quad \text{a.e.} \quad t \geq 0, \tag{4.69}$$

$$y_c(t) = h_c(u_c(t), x_c(t)) + J_c(u_c(t), x_c(t))u_c(t), \tag{4.70}$$

where $x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^l$, $y_c \in \mathbb{R}^m$, $f_c : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$, $G_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_c\times l}$, $h_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^m$, and $J_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^{m\times l}$. We assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, $J(\cdot)$, $f_c(\cdot)$, $G_c(\cdot)$, $h_c(\cdot,\cdot)$, and $J_c(\cdot,\cdot)$ are Lebesgue measurable and locally essentially bounded, (4.69) and (4.70) has at least one equilibrium point, and the required properties for the existence of solutions of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ are satisfied. Note that with the negative feedback interconnection given by Figure 4.1, $u_c = y$ and $y_c = -u$. We assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed, that is, $\det[I_m + J_c(y, x_c)J(x)] \neq 0$ for all y, x, and x_c.

The following results give sufficient conditions for Lyapunov, asymptotic, and exponential stability of the feedback interconnection given by Figure 4.1. In this section, we assume that the forward path $\mathcal{G}$ and the feedback path $\mathcal{G}_c$ in Figure 4.1 are strongly dissipative systems. This assumption holds when the closed-loop system (4.67)–(4.70) admits a unique solution and is only made for notational convenience. Finally, we also note that the obtained stability results also hold for the case where $\mathcal{G}$ and $\mathcal{G}_c$ are weakly dissipative. In this case, however, the set-valued Lie derivative operator should be replaced with the upper right Dini directional derivative in the proofs of the stability theorems.

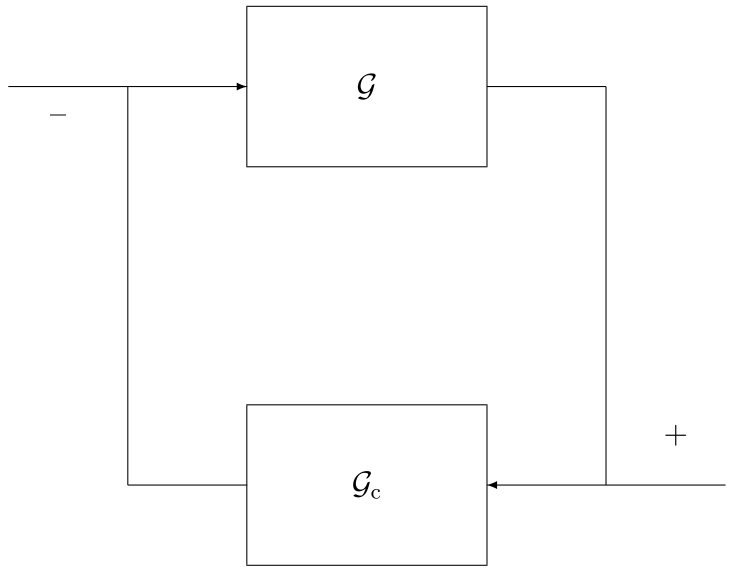

Figure 4.1. Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$.

The following lemma is necessary for the next theorem.

Lemma 4.4.1 [5]. Let $x : [t_0, t] \to \mathbb{R}^q$ be a Filippov solution of the discontinuous dynamical system (3.2) and let $V : \mathbb{R}^q \to \mathbb{R}$ be locally Lipschitz continuous and regular. Then $\frac{\mathrm{d}}{\mathrm{d}\sigma} V(x(\sigma))$ exists for almost all $\sigma \in [t_0, t]$ and $\frac{\mathrm{d}}{\mathrm{d}\sigma} V(x(\sigma)) \in \mathcal{L}_f V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$.

Theorem 4.4.1. Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by (4.67) and (4.68), and $\mathcal{G}_c$ given by (4.69) and (4.70) with input-output pairs (u, y) and (u_c, y_c), respectively, and with $u_c = y$ and $y_c = -u$. Assume $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable, strongly completely reachable, and strongly dissipative with respect to the supply rates $s(u, y)$ and $s_c(u_c, y_c)$ and with locally Lipschitz continuous, regular, and radially unbounded storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that $V_s(0) = 0$ and $V_{sc}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $s(u, y) + \sigma s_c(u_c, y_c) \leq 0$, for all $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $u_c \in \mathbb{R}^l$, $y_c \in \mathbb{R}^m$ such that $u_c = y$ and $y_c = -u$. Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable.

ii) If $\mathcal{G}_c$ is strongly exponentially dissipative with respect to supply rate $s_c(u_c, y_c)$ and $\operatorname{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly asymptotically stable.

iii) If $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially dissipative with respect to supply rates $s(u, y)$ and $s_c(u_c, y_c)$, respectively, and $V_s(\cdot)$ and $V_{sc}(\cdot)$ are such that there exist constants α, α_c, β, and $\beta_c > 0$ such that

$$\alpha\|x\|^2 \le V_s(x) \le \beta\|x\|^2, \qquad x \in \mathbb{R}^n, \tag{4.71}$$

$$\alpha_c\|x_c\|^2 \le V_{sc}(x_c) \le \beta_c\|x_c\|^2, \quad x_c \in \mathbb{R}^{n_c}, \tag{4.72}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly exponentially stable.

Proof. *i*) Note that the closed-loop dynamics of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ has a form given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t), x_c(t)) \\ f_2(x(t), x_c(t)) \end{bmatrix} \triangleq \tilde{f}(x(t), x_c(t)), \quad \begin{bmatrix} x(t_0) \\ x_c(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{c0} \end{bmatrix}, \quad \text{a.e. } t \ge t_0. \tag{4.73}$$

Now, consider the Lyapunov function candidate $V(x, x_c) = V_s(x) + \sigma V_{sc}(x_c)$. Since $\mathcal{L}_{\tilde{f}}V(x, x_c) \subseteq \mathcal{L}_{\tilde{f}}V_s(x) + \sigma\mathcal{L}_{\tilde{f}}V_{sc}(x_c)$ for almost all $(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$, it follows that

$$\begin{aligned} \max \mathcal{L}_{\tilde{f}}V(x, x_c) &\le \max\{\mathcal{L}_{f_1}V_s(x) + \sigma\mathcal{L}_{f_2}V_{sc}(x_c)\} \\ &\le \max \mathcal{L}_{f_1}V_s(x) + \sigma \max \mathcal{L}_{f_2}V_{sc}(x_c). \end{aligned}$$

Next, since $s(u, y) + \sigma s_c(u_c, y_c) \le 0$, for all $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $u_c \in \mathbb{R}^l$, $y_c \in \mathbb{R}^m$, $\frac{\mathrm{d}}{\mathrm{d}t}V_s(x(t)) \in \mathcal{L}_{f_1}V_s(x(t))$, a.e. $t \ge 0$, and $\frac{\mathrm{d}}{\mathrm{d}t}V_{sc}(x_c(t)) \in \mathcal{L}_{f_2}V_{sc}(x_c(t))$, a.e. $t \ge 0$, there exist u', y', u_c' and y_c' such that

$$\max \mathcal{L}_{\tilde{f}} V(x, x_c) \leq \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c) \leq s(u', y') + \sigma s_c(u_c', y_c') \leq 0$$

for almost all $x \in \mathbb{R}^n$ and $x_c \in \mathbb{R}^{n_c}$. Now, it follows from Theorem 2.2.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable.

ii) If $\mathcal{G}_c$ is strongly exponentially dissipative it follows that there exist u', y', u_c' and y_c' and a scalar $\varepsilon_c > 0$ such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) &\leq \max \mathcal{L}_{\tilde{f}} V(x, x_c) \\
&\leq \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c) \\
&\leq -\sigma \varepsilon_c V_{sc}(x_c) + s(u', y') + \sigma s_c(u_c', y_c') \\
&\leq -\sigma \varepsilon_c V_{sc}(x_c), \quad \text{a.e.} \quad (x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}.
\end{aligned}$$

Now, let $\mathcal{R} \triangleq \{(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c} : \frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) = 0 \in \mathcal{L}_{\tilde{f}} V(x, x_c)\}$ and, since $V_{sc}(x_c)$ is positive definite, note that $\frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) = 0$ if and only if $x_c = 0$. Now, since $\mathrm{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, it follows that on every invariant set $\mathcal{M}$ contained in $\mathcal{R}$, $u_c(t) = y(t) \equiv 0$, and hence, by (4.70), $u(t) \equiv 0$ so that $\dot{x}(t) = f(x(t))$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that $\mathcal{M} = \{(0, 0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 2.2.2 that $\mathrm{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (4.73). Now, global strong asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded.

iii) Finally, if $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially dissipative it follows that there exist u', y', u_c' and y_c', and scalars $\varepsilon > 0$ and $\varepsilon_c > 0$ such that

$$\begin{aligned}
\max \mathcal{L}_{\tilde{f}} V(x, x_c) &\leq \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c) \\
&\leq -\varepsilon V_s(x) - \sigma \varepsilon_c V_{sc}(x_c) + s(u', y') + \sigma s_c(u_c', y_c') \\
&\leq -\min\{\varepsilon, \varepsilon_c\} V(x, x_c), \quad (x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}.
\end{aligned}$$

Hence, it follows from Theorem 2.2.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally strongly exponentially stable. $\square$

The next result presents Lyapunov, asymptotic, and exponential stability of dissipative discontinuous feedback systems with supply rates consisting of quadratic supply rates.

Theorem 4.4.2. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, $Q_{\rm c} \in \mathbb{S}^m$, $S_{\rm c} \in \mathbb{R}^{m\times l}$, and $S_{\rm c} \in \mathbb{S}^l$. Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by (4.67) and (4.68) and $\mathcal{G}_{\rm c}$ given by (4.69) and (4.70), and assume $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are strongly zero-state observable. Furthermore, assume $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{\rm s}(\cdot)$, and $\mathcal{G}_{\rm c}$ is strongly dissipative with respect to the supply rate $s_{\rm c}(u_{\rm c}, y_{\rm c}) = y_{\rm c}^{\rm T}Q_{\rm c}y_{\rm c} + 2y_{\rm c}^{\rm T}S_{\rm c}u_{\rm c} + u_{\rm c}^{\rm T}R_{\rm c}u_{\rm c}$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{\rm sc}(\cdot)$. Finally, assume there exists $\sigma > 0$ such that

$$\hat{Q} \triangleq \begin{bmatrix} Q + \sigma R_{\rm c} & -S + \sigma S_{\rm c}^{\rm T} \\ -S^{\rm T} + \sigma S_{\rm c} & R + \sigma Q_{\rm c} \end{bmatrix} \leq 0. \tag{4.74}$$

Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is strongly Lyapunov stable.

ii) If $\mathcal{G}_{\rm c}$ is strongly exponentially dissipative with respect to supply rate $s_{\rm c}(u_{\rm c}, y_{\rm c})$ and

$\operatorname{rank}[G_{\rm c}(u_{\rm c}, 0)] = m$, $u_{\rm c} \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally strongly asymptotically stable.

iii) If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are strongly exponentially dissipative with respect to supply rates $s(u,y)$ and $s_{\rm c}(u_{\rm c}, y_{\rm c})$ and there exist constants $\alpha, \beta, \alpha_{\rm c}$, and $\beta_{\rm c} > 0$ such that (4.71) and (4.72) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally strongly exponentially stable.

iv) If $\hat{Q} < 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly asymptotically stable.

Proof. Statements *i*)–*iii*) are a direct consequence of Theorem 4.4.1 by noting that

$$s(u,y) + \sigma s_c(u_c, y_c) = \begin{bmatrix} y \\ y_c \end{bmatrix}^{\mathrm{T}} \hat{Q} \begin{bmatrix} y \\ y_c \end{bmatrix},$$

and hence, $s(u,y) + \sigma s_c(u_c, y_c) \leq 0$.

To show *iv*) consider the Lyapunov function candidate $V(x, x_c) = V_s(x) + \sigma V_{sc}(x_c)$. Now, since $\mathcal{G}$ and $\mathcal{G}_c$ are strongly dissipative it follows that there exist u', y', u_c' and y_c' with $u_c' = y'$ and $y_c' = -u'$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) &\leq \max \mathcal{L}_{\tilde{f}} V(x, x_c) \\ &\leq \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c) \\ &\leq s(u,y) + \sigma s_c(u_c, y_c) \\ &= y^{\mathrm{T}} Q y + 2y^{\mathrm{T}} S u + u^{\mathrm{T}} R u + \sigma(y_c^{\mathrm{T}} Q_c y_c + 2y_c^{\mathrm{T}} S_c u_c + u_c^{\mathrm{T}} R_c u_c) \\ &= \begin{bmatrix} y \\ y_c \end{bmatrix}^{\mathrm{T}} \hat{Q} \begin{bmatrix} y \\ y_c \end{bmatrix} \\ &\leq 0, \quad \text{a.e.} \quad (x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}, \end{aligned}$$

which implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable. Next, let $\mathcal{R} \triangleq \{(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c} : \frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) = 0 \in \mathcal{L}_{\tilde{f}} V(x, x_c)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) = 0$ if and only if $(y, y_c) = (0, 0)$. Now, since $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable it follows that $\mathcal{M} = \{(0,0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 2.2.2 that $\mathrm{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (4.73). Finally, global strong asymptotic stability follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded, and hence, $V(x, x_c) \to \infty$ as $||(x, x_c)|| \to \infty$. □

The following corollary to Theorem 4.4.2 is necessary for the results in Section 5.4.

Corollary 4.4.1. Consider the closed-loop system consisting of the discontinuous nonlinear dynamical systems $\mathcal{G}$ given by (4.67) and (4.68), and $\mathcal{G}_{\rm c}$ given by (4.69) and (4.70). Let $\alpha, \beta, \alpha_{\rm c}, \beta_{\rm c}, \delta \in \mathbb{R}$ be such that $\beta > 0$, $0 < \alpha+\beta$, $0 < 2\delta < \beta-\alpha$, $\alpha_{\rm c} = \alpha+\delta$, and $\beta_{\rm c} = \beta-\delta$, let $M \in \mathbb{R}^{m\times m}$ be positive definite, and assume $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are strongly zero-state observable. If $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = u^{\rm T}My + \frac{\alpha\beta}{\alpha+\beta}y^{\rm T}My + \frac{1}{\alpha+\beta}u^{\rm T}Mu$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{\rm s}(\cdot)$, and $\mathcal{G}_{\rm c}$ is strongly dissipative with respect to the supply rate $s_{\rm c}(u_{\rm c}, y_{\rm c}) = u_{\rm c}^{\rm T}My_{\rm c} - \frac{1}{\alpha_{\rm c}+\beta_{\rm c}}y_{\rm c}^{\rm T}My_{\rm c} - \frac{\alpha_{\rm c}\beta_{\rm c}}{\alpha_{\rm c}+\beta_{\rm c}}u_{\rm c}^{\rm T}Mu_{\rm c}$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{\rm sc}(\cdot)$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally strongly asymptotically stable.

Proof. The proof is a direct consequence of Theorem 4.4.2 with $Q = \frac{\alpha\beta}{\alpha+\beta}M$, $S = \frac{1}{2}M$, $R = \frac{1}{\alpha+\beta}M$, $Q_{\rm c} = -\frac{1}{\alpha_{\rm c}+\beta_{\rm c}}M$, $S_{\rm c} = \frac{1}{2}M$, and $R_{\rm c} = -\frac{\alpha_{\rm c}\beta_{\rm c}}{\alpha_{\rm c}+\beta_{\rm c}}M$. Specifically, let $\sigma > 0$ be such that

$$\sigma\left(\frac{\delta^2}{(\alpha+\beta)^2} - \frac{1}{4}\right) + \frac{1}{4} > 0.$$

In this case, $\hat{Q}$ given by (4.74) satisfies

$$\hat{Q} = \begin{bmatrix} (\frac{\alpha\beta}{\alpha+\beta} - \frac{\sigma\alpha_{\rm c}\beta_{\rm c}}{\alpha_{\rm c}+\beta_{\rm c}})M & \frac{\sigma-1}{2}M \\ \frac{\sigma-1}{2}M & (\frac{1}{\alpha+\beta} - \frac{\sigma}{\alpha_{\rm c}+\beta_{\rm c}})M \end{bmatrix} < 0,$$

so that all the conditions of Theorem 4.4.2 are satisfied. □

The following corollary is a direct consequence of Theorem 4.4.2. Note that if a nonlinear discontinuous dynamical system $\mathcal{G}$ is strongly dissipative with respect to a supply rate $s(u,y) = u^{\rm T}y - \varepsilon u^{\rm T}u - \hat{\varepsilon}y^{\rm T}y$, where $\varepsilon, \hat{\varepsilon} \geq 0$, then with $\kappa(y) = ky$, where $k \in \mathbb{R}$ is such that $k(1-\varepsilon k) < \hat{\varepsilon}$, $s(u,y) = [k(1-\varepsilon k) - \hat{\varepsilon}]y^{\rm T}y < 0$, $y \neq 0$. Hence, if $\mathcal{G}$

is strongly zero-state observable it follows from Theorem 3.2 of [38] that all storage functions of $\mathcal{G}$ are positive definite.

Corollary 4.4.2. Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by (4.67) and (4.68) and $\mathcal{G}_\mathrm{c}$ given by (4.69) and (4.70), and assume $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly zero-state observable. Then the following statements hold:

i) If $\mathcal{G}$ is strongly passive, $\mathcal{G}_\mathrm{c}$ is strongly exponentially passive, and $\mathrm{rank}[G_\mathrm{c}(u_\mathrm{c}, 0)] = m$, $u_\mathrm{c} \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is strongly asymptotically stable.

ii) If $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly exponentially passive with storage functions $V_\mathrm{s}(\cdot)$ and $V_\mathrm{sc}(\cdot)$, respectively, such that (4.71) and (4.72) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is strongly exponentially stable.

iii) If $\mathcal{G}$ is strongly nonexpansive with gain $\gamma > 0$, $\mathcal{G}_\mathrm{c}$ is strongly exponentially nonexpansive with gain $\gamma_\mathrm{c} > 0$, $\mathrm{rank}[G_\mathrm{c}(u_\mathrm{c}, 0)] = m$, $u_\mathrm{c} \in \mathbb{R}^l$, and $\gamma\gamma_\mathrm{c} \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is strongly asymptotically stable.

iv) If $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly exponentially nonexpansive with storage functions $V_\mathrm{s}(\cdot)$ and $V_\mathrm{sc}(\cdot)$, respectively, such that (4.71) and (4.72) hold, and with gains $\gamma > 0$ and $\gamma_\mathrm{c} > 0$, respectively, such that $\gamma\gamma_\mathrm{c} \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is strongly exponentially stable.

Proof. The proof is a direct consequence of Theorem 4.4.2. Specifically, *i*) and *ii*) follow from Theorem 4.4.2 with $Q = Q_\mathrm{c} = 0$, $S = S_\mathrm{c} = I_m$, and $R = R_\mathrm{c} = 0$, whereas *iii*) and *iv*) follow from Theorem 4.4.2 with $Q = -I_l$, $S = 0$, $R = \gamma^2 I_m$, $Q_\mathrm{c} = -I_{l_\mathrm{c}}$, $S_\mathrm{c} = 0$, and $R_\mathrm{c} = \gamma_\mathrm{c}^2 I_{m_\mathrm{c}}$. □

Example 4.4.1. Consider the nonlinear mechanical system $\mathcal{G}$ with a discontinuous spring force given by

$$\ddot{x}(t) + \operatorname{sign}(x(t)) = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad \text{a.e.} \quad t \geq 0, \tag{4.75}$$

$$y(t) = \frac{1}{2}\dot{x}(t), \tag{4.76}$$

or, equivalently,

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \tag{4.77}$$

$$\dot{x}_2(t) = -\operatorname{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20}, \tag{4.78}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{4.79}$$

and the continuous nonlinear second-order dynamic controller $\mathcal{G}_\mathrm{c}$ given by

$$\dot{x}_{\mathrm{c}1}(t) = -\frac{1}{2}x_{\mathrm{c}1}(t) - x_{\mathrm{c}2}(t), \quad x_{\mathrm{c}1}(0) = x_{\mathrm{c}10}, \quad t \geq 0, \tag{4.80}$$

$$\dot{x}_{\mathrm{c}2}(t) = -10x_{\mathrm{c}1}^3(t) - 10x_{\mathrm{c}2}(t) + 5u_\mathrm{c}(t), \quad x_{\mathrm{c}2}(0) = x_{\mathrm{c}20}, \tag{4.81}$$

$$y_\mathrm{c}(t) = 10x_{\mathrm{c}2}(t). \tag{4.82}$$

Furthermore, consider the feedback interconnection of (4.77)–(4.82) given by $u = -y_\mathrm{c}$ and $u_\mathrm{c} = y$. Next, let $V_\mathrm{s}(x) = |x_1| + \frac{1}{2}x_2^2$ and note that, for almost all $x \in \mathbb{R}^2$,

$$\partial V_\mathrm{s}(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{x_2\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, \\ [-1, 1] \times \{x_2\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0. \end{cases}$$

Hence, $\mathcal{L}_f V_\mathrm{s}(x_1, x_2) = \{0\}$ and $\mathcal{L}_G V_\mathrm{s}(x_1, x_2) = \{x_2\}$, which implies that $\min \mathcal{L}_f V_\mathrm{s}(x) = \max \mathcal{L}_f V_s(x) = 0$ for almost all $x \in \mathbb{R}^2$. Now, with $\varepsilon = 0$, $\ell(x) = 0$, and $\mathcal{W}(x) = 0$, (4.37)–(4.40) are satisfied. Hence, it follows from Corollary 4.3.1 that $\mathcal{G}$ is weakly passive.

Next, note that with $\mathcal{W}(x_\mathrm{c}) \equiv 0$, $\ell(x_\mathrm{c}) = \pm\sqrt{10x_{\mathrm{c}1}^4(2-\varepsilon) + 2x_{\mathrm{c}2}^2(20-\varepsilon)}$, $V_{\mathrm{sc}}(x_\mathrm{c}) = 10x_{\mathrm{c}1}^4 + 2x_{\mathrm{c}2}^2$, and $\varepsilon \in (0, 2]$, it follows from Corollary 4.3.1 that $\mathcal{G}_\mathrm{c}$ is exponentially passive. Furthermore, $\operatorname{rank}[G_\mathrm{c}(u_\mathrm{c}, 0)] = 1$, $u_\mathrm{c} \in \mathbb{R}$. Now, it follows from *ii)*

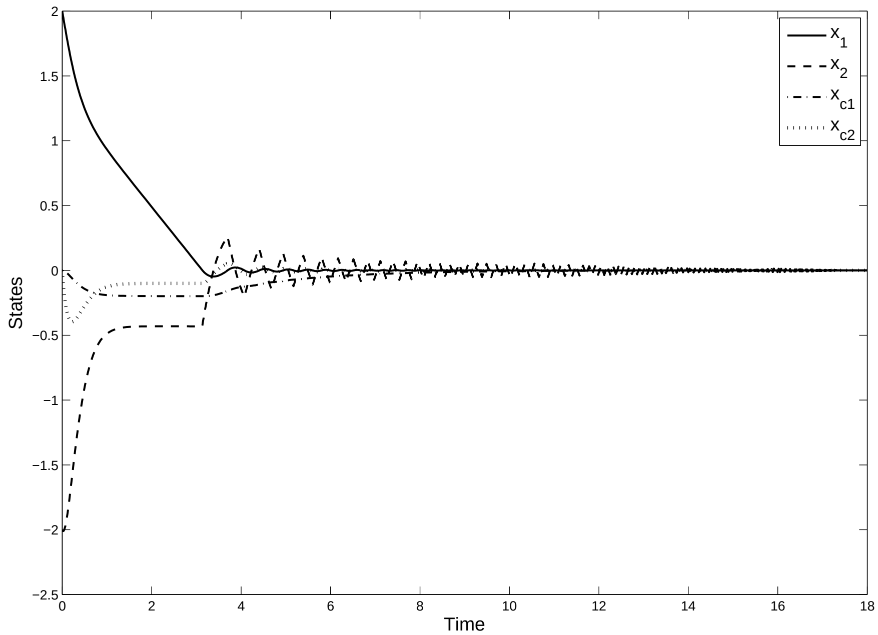

Figure 4.2. State trajectories of the closed-loop system versus time for the full-order controller.

of Theorem 4.4.2 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally asymptotically stable. Figure 4.2 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\rm T}$ and $x_{\rm c}(0) = 0$.

Alternatively, we consider the reduced-order dynamic controller $\mathcal{G}_{\rm c}$ given by

$$\dot{x}_{\rm c}(t) = -10x_{\rm c}(t) + 20u_{\rm c}(t), \quad x_{\rm c}(0) = x_{\rm c0}, \quad t \geq 0, \tag{4.83}$$

$$y_{\rm c}(t) = 12x_{\rm c}(t). \tag{4.84}$$

Note that with $V_{\rm sc}(x_{\rm c}) = \frac{3}{5}x_{\rm c}^2$, $\varepsilon = 20$, $\ell(x_{\rm c}) \equiv 0$, and $\mathcal{W}(x_{\rm c}) \equiv 0$, it follows from Corollary 4.3.1 that $\mathcal{G}_{\rm c}$ is exponentially passive. Moreover, $\operatorname{rank}[G_{\rm c}(u_{\rm c}, 0)] = 1$, $u_{\rm c} \in \mathbb{R}$. Hence, it follows from *ii*) of Theorem 4.4.2 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally asymptotically stable. Figure 4.3 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\rm T}$ and $x_{\rm c}(0) = 0$. △

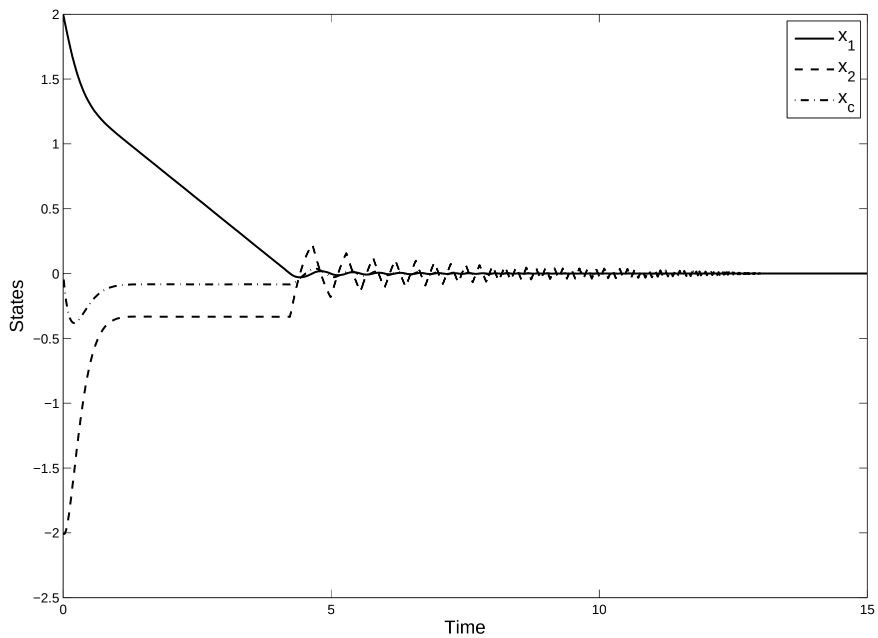

Figure 4.3. State trajectories of the closed-loop system versus time for the reduced-order controller.

Chapter 5

On the Equivalence Between Dissipativity and Optimality of Discontinuous Nonlinear Regulators for Filippov Systems

5.1. Introduction

For continuous-time nonlinear dynamical systems with continuously differentiable flows, the problem of guaranteed stability margins for optimal and inverse optimal regulators is well known [26, 51, 52]. Specifically, nonlinear inverse optimal controllers that minimize a (in the terminology of [26]) *meaningful* nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic, nonnegative-definite function of the state and a quadratic positive definite function of the control are known to possess sector margin guarantees to component decoupled memoryless input nonlinearities lying in the conic sector $(\frac{1}{2}, \infty)$. These results also hold for disk margin guarantees where asymptotic stability of the closed-loop system is guaranteed in the face of a dissipative dynamic input operator. In addition, using a certain return difference condition, closely related to loop gain concepts in linear systems theory, an equivalence between dissipativity with respect to a quadratic supply rate and optimality of a nonlinear feedback regulator also holds [52].

In a two-part paper [32,33], the authors extend the results of [26,51,52] to develop a general framework for hybrid feedback systems by addressing stability, dissipativity, optimality, and inverse optimality of impulsive dynamical systems. In particular, [33] considers a hybrid feedback optimal control problem over an infinite horizon involving a hybrid nonlinear-nonquadratic performance functional. In [34], sufficient conditions for hybrid gain, sector, and disk margins guarantees for nonlinear hybrid dynamical systems were developed. In [7], the authors provide a sufficient condition for discontinuous $\mathcal{L}_2$-gain stabilizability of a nonlinear affine system with respect to Filippov solutions. Their sufficient condition requires the existence of a viscosity supersolution of a Hamiton–Jacobi–Bellman equation.

In this chapter, we develop sufficient conditions for gain, sector, and disk margins guarantees for Filippov nonlinear dynamical systems controlled by optimal and inverse optimal discontinuous regulators. Furthermore, we develop a counterpart to the classical return difference inequality for continuous-time systems with continuously differentiable flows [11, 52] for Filippov dynamical systems and provide connections between dissipativity and optimality for discontinuous nonlinear controllers. In particular, we show an equivalence between dissipativity and optimality of discontinuous controllers holds for Filippov dynamical systems. Specifically, we show that an optimal nonlinear controller $\phi(x)$ satisfying a return difference condition is equivalent to the fact that the Filippov dynamical system with input u and output $y = -\phi(x)$ is dissipative with respect to a supply rate of the form $[u + y]^{\mathrm{T}}[u + y] - u^{\mathrm{T}}u$.

5.2. Stability Margins for Discontinuous Feedback Regulators

To develop relative stability margins for discontinuous nonlinear regulators consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{5.1}$$

$$y(t) = -\phi(x(t)), \tag{5.2}$$

where $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded, and $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is a discontinuous feedback controller such that $\mathcal{G}$ is weakly (resp., strongly) asymptotically stable with $u = -y$. Furthermore, assume that the system $\mathcal{G}$ is weakly (resp., strongly) zero-state observable. Next, we define the relative stability margins for $\mathcal{G}$ given by (5.1) and (5.2). Specifically, let $u_c \triangleq -y$, $y_c \triangleq u$, and consider the negative feedback interconnection $u = \Delta(-y)$ of $\mathcal{G}$ and $\Delta(\cdot)$ given in Figure 5.1, where $\Delta(\cdot)$ is either a linear operator $\Delta(u_c) = \Delta u_c$, a nonlinear static operator $\Delta(u_c) = \sigma(u_c)$, or a dynamic nonlinear operator $\Delta(\cdot)$ with input u_c and output y_c. Furthermore, we assume that in the nominal case $\Delta(\cdot) = I(\cdot)$ so that the nominal closed-loop system is weakly (resp., strongly) asymptotically stable.

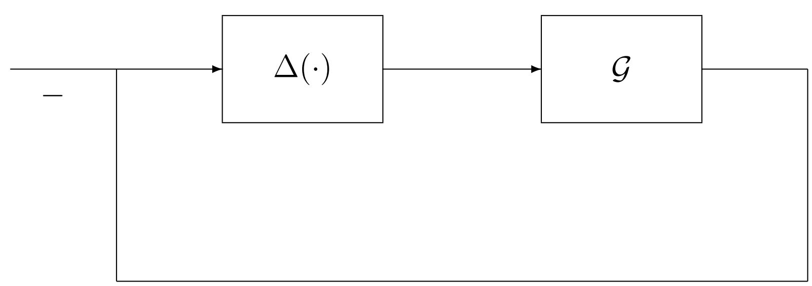

Figure 5.1. Multiplicative input uncertainty of $\mathcal{G}$ and input operator $\Delta(\cdot)$.

Definition 5.2.1. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.2) is said to have a *weak* (resp., *strong*) *gain margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_c) = \Delta u_c$ is globally weakly (resp., strongly) asymptotically stable for all $\Delta = \mathrm{diag}[k_1, \ldots, k_m]$, where $k_i \in (\alpha, \beta)$, $i = 1, \ldots, m$.

Definition 5.2.2. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.2) is said to have a *weak* (resp., *strong*) *sector margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_{\rm c}) = \sigma(u_{\rm c})$ is globally weakly (resp., strongly) asymptotically stable for all nonlinearities $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ such that $\sigma(0) = 0$, $\sigma(u_{\rm c}) = [\sigma_1(u_{\rm c1}), \ldots, \sigma_m(u_{{\rm c}m})]^{\rm T}$, and $\alpha u_{{\rm c}i}^2 < \sigma_i(u_{{\rm c}i})u_{{\rm c}i} < \beta u_{{\rm c}i}^2$, for all $u_{{\rm c}i} \neq 0$, $i = 1, \ldots, m$.

Definition 5.2.3. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.2) is said to have a *weak* (resp., *strong*) *disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally weakly (resp., strongly) asymptotically stable for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is weakly (resp., strongly) zero-state observable and weakly (resp., strongly) dissipative with respect to the supply rate $s(u_{\rm c}, y_{\rm c}) = u_{\rm c}^{\rm T} y_{\rm c} - \frac{1}{\hat{\alpha}+\hat{\beta}} y_{\rm c}^{\rm T} y_{\rm c} - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}} u_{\rm c}^{\rm T} u_{\rm c}$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Definition 5.2.4. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.2) is said to have a *weak* (resp., *strong*) *structured disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally weakly (resp., strongly) asymptotically stable for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is weakly (resp., strongly) zero-state observable, $\Delta(u_{\rm c}) = {\rm diag}[\delta_1(u_{\rm c1}), \ldots, \delta_m(u_{{\rm c}m})]$, and $\delta_i(\cdot)$, $i = 1, \ldots, m$, is weakly (resp., strongly) dissipative with respect to the supply rate $s(u_{{\rm c}i}, y_{{\rm c}i}) = u_{{\rm c}i} y_{{\rm c}i} - \frac{1}{\hat{\alpha}+\hat{\beta}} y_{{\rm c}i}^2 - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}} u_{{\rm c}i}^2$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Remark 5.2.1. Note that if $\mathcal{G}$ has a weak (resp., strong) disk margin (α, β), then $\mathcal{G}$ has weak (resp., strong) gain and sector margins (α, β).

5.3. Nonlinear-Nonquadratic Optimal Regulators for Discontinuous Dynamical Systems

In this section, we consider a control problem involving a notion of optimality with respect to a nonlinear-nonquadratic cost functional. To address the optimal control problem let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set and let $U \subseteq \mathbb{R}^m$, where $0 \in \mathcal{D}$ and $0 \in U$. Next, consider the controlled nonlinear discontinuous dynamical system (3.1), where $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$ for almost all $t \geq 0$ and the constraint set U is given. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the closed-loop dynamical system shown in the Figure 5.2 is given by (3.2).

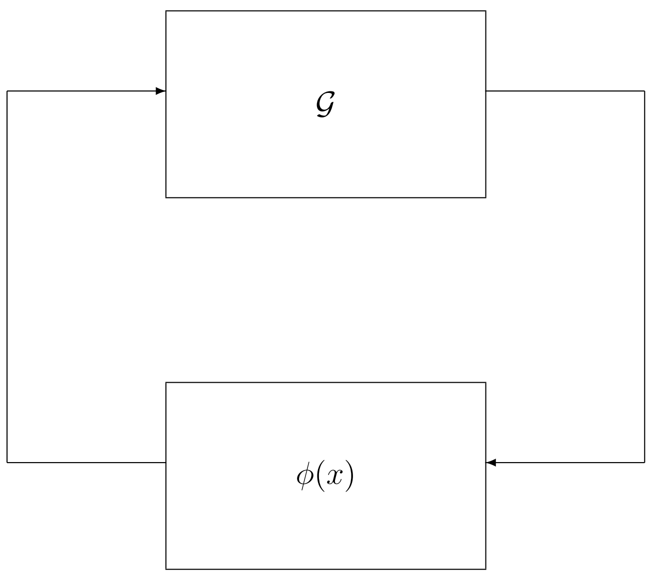

Figure 5.2. Nonlinear closed-loop feedback system.

Next, we present a main theorem for characterizing feedback controllers that guarantee stability of the controlled discontinuous dynamical system $\mathcal{G}$ and minimize a nonlinear-nonquadratic performance functional. For the statement of this result let $L : \mathcal{D} \times U \to \mathbb{R}$ be Lipschitz continuous and define the set of regulation controllers by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) \in U : u(\cdot) \text{ is measurable and locally essentially bounded,}$$
$$\text{and } x(\cdot) \text{ driven by (2.1) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_0)$, that is, inputs corresponding to null convergent solutions, can be interpreted as incorporating a system detectability condition through the cost.

Theorem 5.3.1. Consider the controlled discontinuous nonlinear dynamical system (3.1) with performance functional[5]

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t, \tag{5.3}$$

where (5.3) is defined with respect to absolutely continuous state arcs $x(\cdot)$ and measurable control functions $u : [0, \infty) \to U$. Assume that there exists a locally Lipschitz continuous and regular function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$V(0) = 0, \tag{5.4}$$
$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{5.5}$$
$$\phi(0) = 0, \tag{5.6}$$
$$\max \mathcal{L}_{F(\cdot,\phi(\cdot))}V(x) < 0, \quad \text{a.e.} \quad x \in \mathcal{D}, \quad x \neq 0, \tag{5.7}$$
$$\mathcal{H}(x, \phi(x)) = 0, \quad \text{a.e.} \quad x \in \mathcal{D}, \tag{5.8}$$
$$\mathcal{H}(x, u) \geq 0, \quad \text{a.e.} \quad x \in \mathcal{D}, \quad u \in U, \tag{5.9}$$

where

$$\mathcal{H}(x, u) \triangleq L(x, u) + \min \mathcal{L}_{F(\cdot,u)}V(x). \tag{5.10}$$

[5]Since solutions to (3.1) are not necessarily unique, $J(x_0, u(\cdot))$ given by (5.3) depends on the particular state trajectory $x(\cdot)$ along which we integrate. Alternatively, if we assume that $f(\cdot, u)$ is essentially one-sided Lipschitz on $\mathcal{B}_\delta(x)$ for some $\delta > 0$, then there exists a unique Filippov solution to (3.1) with initial condition $x(t_0) = x_0$ and $u(t) \in U$ [24].

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the zero Filippov solution $x(t) \equiv 0$ of the closed-loop system (3.2) is locally strongly asymptotically stable and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathcal{D}_0. \tag{5.11}$$

In addition, if $x_0 \in \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{5.12}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and

$$V(x) \to \infty \text{ as } ||x|| \to \infty, \tag{5.13}$$

then the zero Filippov solution $x(t) \equiv 0$ of the closed-loop system (3.2) is globally strongly asymptotically stable.

Proof. Local and global strong asymptotic stability follow from (5.4)–(5.7) by applying Theorem 2.2.1 to the closed-loop system (3.2). Next, with $u(t) \equiv \bar{u}(t)$, where $\bar{u}(\cdot)$ is measurable and locally essentially bounded, let $\bar{\psi}(t)$, $t \geq 0$, be any Filippov solution of (3.1). Then, it follows that $\mathcal{L}_{F(\cdot,\bar{u}(\cdot))} V(\bar{\psi}(t)) \subseteq \mathcal{L}_{F(\cdot,u)} V(\bar{\psi}(t))$ for almost every $t \geq 0$. Moreover, it follows from Lemma 4.2.1 that $\frac{\mathrm{d}}{\mathrm{d}t} V(\bar{\psi}(t)) \in \mathcal{L}_{F(\cdot,\bar{u}(\cdot))} V(\bar{\psi}(t))$ for almost every $t \geq 0$. Now, since $\bar{u}(t)$ and $\bar{\psi}(t)$ are arbitrary, it follows that

$$\min \mathcal{L}_{F(\cdot,u)} V(x(\sigma)) \leq \frac{\mathrm{d}}{\mathrm{d}\sigma} V(x(\sigma)) \leq \max \mathcal{L}_{F(\cdot,u)} V(x(\sigma)), \\ \text{a.e.} \quad \sigma \in [0, t], \quad u \in U. \tag{5.14}$$

Next, let $x_0 \in \mathcal{D}_0$, let $u(\cdot) \in \mathcal{S}(x_0)$, and let $x(t)$ for almost all $t \geq 0$ be the Filippov solution of (2.1). Then, it follows from (5.14) that

$$L(x(t), u(t)) \geq -\dot{V}(x(t)) + L(x(t), u(t)) + \min \mathcal{L}_{F(\cdot,u)} V(x(t))$$

$$= -\dot{V}(x(t)) + \mathcal{H}(x(t), u(t)), \quad \text{a.e.} \quad t \geq 0. \tag{5.15}$$

Furthermore, note that

$$V(x(t)) = V(x(t_0)) + \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma} V(x(\sigma))\mathrm{d}\sigma, \tag{5.16}$$

where the integral in (5.16) is the Lebesgue integral. Now, using (5.9), (5.15), (5.16), and the fact that $u(\cdot) \in \mathcal{S}(x_0)$, it follows that

$$\begin{aligned} J(x_0, u(\cdot)) &\geq \int_0^\infty [-\dot{V}(x(t)) + \mathcal{H}(x(t), u(t))]\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) + \int_0^\infty \mathcal{H}(x(t), u(t))\mathrm{d}t \\ &= V(x_0) + \int_0^\infty \mathcal{H}(x(t), u(t))\mathrm{d}t \\ &\geq V(x_0) \\ &= J(x_0, \phi(x(\cdot)), \end{aligned}$$

which yields (5.12). □

Note that (5.8) is the steady-state Hamilton-Jacobi-Bellman equation for the discontinuous dynamical system (3.1) with the cost $J(x_0, u(\cdot))$. Since we are not imposing that solutions to (5.8) be smooth, the Hamilton–Jacobi–Bellman equation (5.8) should be interpreted in the viscosity sense (i.e., a viscosity supersolution) [25, 30] or, equivalently, as in the proximal analysis formalism of [14]. Specifically, since $\underline{\partial}V(x) \subseteq \partial V(x)$, where

$$\underline{\partial}V(x) \triangleq \left\{ p \in \mathbb{R}^n : \liminf_{\|h\|\to 0} \frac{V(x+h) - V(x) - p^{\mathrm{T}}h}{\|h\|} \geq 0 \right\},$$

denotes the subdifferential of $V(\cdot)$ at x [13,14], it follows from (5.9) that $V(x)$ is a viscosity supersolution of (5.8). However, in general, $V(x)$ is not a viscosity subsolution of (5.8), which shows that the equivalence between optimal regulation, solvability of the Hamilton-Jacobi-Bellman equation, and feedback stabilizability breaks down for nonsmooth value functions $V(\cdot)$. It is important to note that Theorem 5.3.1 provides constructive *sufficient* conditions for optimality of a feedback controller. Furthermore, this controller is stabilizing and its optimality is independent of the system initial condition x_0. Finally, *necessary* conditions for optimality of nonsmooth regulation and existence of viscosity solutions of the resulting Hamilton-Jacobi-Bellman equation are discussed in [6, 18].

Next, we specialize Theorem 5.3.1 to discontinuous affine dynamical systems. Specifically, we construct discontinuous nonlinear feedback controllers using an optimal control framework that minimizes a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the total generalized derivative of the Lyapunov function is negative along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing viscosity supersolutions to the Hamilton-Jacobi-Bellman equation. Thus, these results provide a family of globally stabilizing controllers parameterized by the cost functional that is minimized.

The controllers obtained in this section are predicated on an *inverse optimal control problem* [26, 31]. In particular, to avoid the complexity in solving the steady-state Hamilton-Jacobi-Bellman equation we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law.

Consider the discontinuous nonlinear affine dynamical system given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{5.17}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. We assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. Furthermore, we consider performance integrands $L(x, u)$ of the form

$$L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u, \tag{5.18}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{P}^m$ with $\mathbb{P}^m$ denoting the set of $m \times m$ positive definite matrices, so that (5.3) becomes

$$J(x_0, u(\cdot)) = \int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t. \quad (5.19)$$

Theorem 5.3.2. Consider the discontinuous nonlinear controlled affine dynamical system (5.17) with performance functional (5.19). Assume that there exists a locally Lipschitz continuous and regular function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \quad (5.20)$$

$$L_2(0) = 0, \quad (5.21)$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (5.22)$$

$$\max \mathcal{L}_{[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)]} V(x) < 0, \ \text{ a.e. } x \in \mathbb{R}^n, \ x \neq 0, \quad (5.23)$$

and

$$V(x) \to \infty \text{ as } ||x|| \to \infty. \quad (5.24)$$

Then the zero Filippov solution $x(t) \equiv 0$ of the closed-loop discontinuous dynamical system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \quad (5.25)$$

is globally strongly asymptotically stable with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}}, \quad (5.26)$$

and the performance functional (5.19), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - \min \mathcal{L}_f V(x), \quad (5.27)$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n. \quad (5.28)$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{5.29}$$

Proof. The result is a direct consequence of Theorem 5.3.1 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u$, and $f(x, u) = f(x) + G(x)u$. Specifically, with (5.18) the Hamiltonian has the form

$$\mathcal{H}(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u + \min \mathcal{L}_f V(x) + \mathcal{L}_G V(x)u.$$

Now, the feedback control law (5.26) is obtained by setting $\frac{\partial \mathcal{H}}{\partial u} = 0$. With (5.26), it follows that (5.20), (5.22), (5.23), and (5.24) imply (5.4), (5.5), (5.7), and (5.13), respectively. Next, since $V(\cdot)$ is locally Lipschitz continuous and regular, and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $\mathcal{L}_G V(0) = 0$, and hence, since by assumption $L_2(0) = 0$, it follows that $\phi(0) = 0$, which implies (5.6). Next, with $L_1(x)$ given by (5.27) and $\phi(x)$ given by (5.26), (5.8) holds. Finally, since $\mathcal{H}(x, u) = \mathcal{H}(x, u) - \mathcal{H}(x, \phi(x)) = [u - \phi(x)]^{\mathrm{T}} R_2(x)[u - \phi(x)]$ and $R_2(x)$ is positive definite for almost all $x \in \mathbb{R}^n$, condition (5.9) holds. The result now follows as a direct consequence of Theorem 5.3.1. □

Example 5.3.1. To illustrate the utility of Theorem 5.3.2 we consider a controlled nonsmooth harmonic oscillator with nonsmooth friction given by ([5])

$$\dot{x}_1(t) = -\operatorname{sign}(x_2(t)) - \frac{1}{2}\operatorname{sign}(x_1(t)), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \tag{5.30}$$

$$\dot{x}_2(t) = \operatorname{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20}, \tag{5.31}$$

where $\operatorname{sign}(\sigma) \triangleq \frac{\sigma}{|\sigma|}$, $\sigma \neq 0$, and $\operatorname{sign}(0) \triangleq 0$. To construct an inverse optimal globally stabilizing control law for (5.30) and (4.52) let $V(x) = |x_1| + |x_2|$ and note that

$$\partial V(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \overline{\operatorname{co}}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V(x_1,x_2) = \begin{cases} \{-\frac{1}{2}\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1,x_2) = (0,0), \end{cases}$$

$$\mathcal{L}_G V(x_1,x_2) = \begin{cases} \{\mathrm{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{\mathrm{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1,x_2) = (0,0), \end{cases}$$

which implies that $\max \mathcal{L}_f V(x) = 0$, $\min \mathcal{L}_f V(x) = -\frac{1}{2}$, and $\mathcal{L}_G V(x) = \{\mathrm{sign}(x_2)\}$ for almost all $x \in \mathbb{R}^2$.

Next, it follows that

$$\mathcal{L}_{\tilde{f}} V(x_1,x_2) = \begin{cases} \{-1-\frac{1}{2}L_2(x_1,x_2)\,\mathrm{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{-1-\frac{1}{2}L_2(x_1,x_2)\,\mathrm{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1,x_2) = (0,0), \end{cases}$$

where $\tilde{f} \triangleq f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)$ with $R_2(x) \equiv 1$. Let $L(x,u) = L_1(x) + L_2(x)u + u^2$. Now, $L_2(x) = x_2$ satisfies (5.23) so that the inverse optimal control law (5.26) is given by

$$\phi(x) = -\frac{1}{2}[\mathrm{sign}(x_2) + x_2], \quad \text{a.e.} \quad x \in \mathbb{R}^2. \tag{5.32}$$

In this case, the performance functional (5.19), with

$$L_1(x) = \frac{1}{4}[\mathrm{sign}(x_2) + x_2]^2 + \frac{1}{2}, \quad \text{a.e.} \quad x \in \mathbb{R}^2, \tag{5.33}$$

is minimized in the sense of (5.28). Furthermore, using the feedback control law (5.32) it follows that

$$\mathcal{L}_{\tilde{f}} V(x_1,x_2) = \begin{cases} \{-1-\frac{1}{2}|x_2|\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{-1-\frac{1}{2}|x_2|\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1,x_2) = (0,0). \end{cases}$$

Note that $\max \mathcal{L}_{\tilde{f}} V(x) \leq 0$. Now, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^2 : \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0 \in \mathcal{L}_{\tilde{f}} V(x)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$ if and only if $x = 0$. Hence, since $\mathcal{M} = \{(0,0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$, it follows from Theorem 2.2.2 that $\operatorname{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (5.30) and (4.52). Now, since $V(x)$ is radially unbounded, the feedback control law (5.32) is globally strongly stabilizing. △

5.4. Gain, Sector, and Disk Margins of Nonlinear-Nonquadratic Optimal Regulators for Discontinuous Systems

In this section, we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion for discontinuous dynamical systems. Specifically, sufficient conditions that guarantee gain, sector, and disk margins are given in terms of the state, control, and cross-weighting nonlinear-nonquadratic weighting functions. In particular, we consider the discontinuous nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{5.34}$$

$$y(t) = -\phi(x(t)), \tag{5.35}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $\phi : \mathbb{R}^n \to \mathbb{R}^m$, with a nonquadratic performance criterion

$$J(x_0, u(\cdot)) = \int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t, \tag{5.36}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are given such that $R_2(x) > 0$, $x \in \mathbb{R}^n$, and $L_2(0) = 0$. Once again, we assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. In this case, the optimal nonlinear feedback controller $u = \phi(x)$ that minimizes the nonlinear-nonquadratic performance criterion (5.36) is given by the following result.

Theorem 5.4.1. Consider the discontinuous nonlinear dynamical system given by (5.34) and (5.35) with performance functional (5.36). Assume that there exists a locally Lipschitz continuous and regular function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{5.37}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \ x \neq 0, \tag{5.38}$$

$$L_2(0) = 0, \tag{5.39}$$

$$\max \mathcal{L}_{[f(x)-\frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x)-\frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)]} V(x) < 0,$$
$$\text{a.e. } x \in \mathbb{R}^n, \ x \neq 0, \tag{5.40}$$

$$L_1(x) + \min \mathcal{L}_f V(x) - \tfrac{1}{4}[\mathcal{L}_G V(x) + L_2(x)]R_2^{-1}(x)[\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}} = 0,$$
$$\text{a.e. } x \in \mathbb{R}^n, \tag{5.41}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{5.42}$$

Then the zero Filippov solution $x(t) \equiv 0$ of the closed-loop discontinuous dynamical system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{5.43}$$

is globally strongly asymptotically stable with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}}, \tag{5.44}$$

and the performance functional (5.36) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n. \tag{5.45}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{5.46}$$

Proof. The proof is a direct consequence of Theorem 5.3.1. □

The following key lemma is needed.

Lemma 5.4.1. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.34) and (5.35) where $\phi(x)$ is a strongly stabilizing feedback control law given by (5.44). Suppose $V(x)$, $x \in \mathbb{R}^n$, satisfies

$$0 = \min \mathcal{L}_f V(x) + L_1(x) - \tfrac{1}{4}[\mathcal{L}_G V(x) + L_2(x)] R_2^{-1}(x) [\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}}, \quad (5.47)$$

$$[\max \mathcal{L}_f V(x) - \min \mathcal{L}_f V(x)] \le L_1(x) - \tfrac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x), \quad \text{a.e.} \quad x \in \mathbb{R}^n, \quad (5.48)$$

with $\theta \in \mathbb{R}$ such that $0 < \theta < 1$. Then, for almost all $u(t) \in U$ and $t_1, t_2 \ge 0$, $t_1 < t_2$, the solution $x(t)$, $t \ge 0$, to (5.34) satisfies

$$V(x(t_2)) \le \int_{t_1}^{t_2} \big\{[u(t)+y(t)]^{\mathrm{T}} R_2(x(t))[u(t)+y(t)] - \theta^2 u^{\mathrm{T}}(t) R_2(x(t)) u(t)\big\} \mathrm{d}t + V(x(t_1)). \quad (5.49)$$

Proof. Note that it follows from (5.44), (5.47), and (5.48) that for almost all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\begin{aligned}
\theta^2 u^{\mathrm{T}} R_2(x) u \le\ & \theta^2 u^{\mathrm{T}} R_2(x) u + \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right] \\
& \times R_2(x) \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right]^{\mathrm{T}} \\
=\ & u^{\mathrm{T}} R_2(x) u + \frac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x) + L_2(x) u \\
\le\ & u^{\mathrm{T}} R_2(x) u + L_2(x) u + L_1(x) - [\max \mathcal{L}_f V(x) - \min \mathcal{L}_f V(x)] \\
=\ & u^{\mathrm{T}} R_2(x) u + [L_2(x) + \mathcal{L}_G V(x)] u + \min \mathcal{L}_f V(x) - \min \mathcal{L}_f V(x) \\
& + \phi^{\mathrm{T}}(x) R_2(x) \phi(x) - \max \mathcal{L}_f V(x) - \mathcal{L}_G V(x) u \\
=\ & [u+y]^{\mathrm{T}} R_2(x)[u+y] - \max \mathcal{L}_f V(x) - \mathcal{L}_G V(x) u. \qquad (5.50)
\end{aligned}$$

Next, using the sum rule for the generalized gradient of locally Lipschitz continuous functions [56] it follows that $\mathcal{L}_{f+Gu} V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_{Gu} V(x)$ for almost all $x \in \mathbb{R}^n$. Now, it follows from Lemma 4.2.1 that $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \in \mathcal{L}_{f+Gu} V(x(t)) \subseteq$

$\mathcal{L}_f V(x(t)) + \mathcal{L}_{Gu} V(x(t))$ for almost all $t \geq 0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &\leq \max \mathcal{L}_{f+Gu} V(x(t)) \\ &\leq \max \left[\mathcal{L}_f V(x(t)) + \mathcal{L}_G V(x(t)) u(t)\right] \\ &= \max \mathcal{L}_f V(x(t)) + \mathcal{L}_G V(x(t)) u(t), \quad \text{a.e. } t \geq 0, \;\; u(t) \in U. \quad (5.51)\end{aligned}$$

It follows from (5.50) and (5.51) that, for all $u(t) \in U$ and almost all $t \geq 0$,

$$\theta^2 u^{\mathrm{T}}(t) R_2(x(t)) u(t) \leq [u(t) + y(t)]^{\mathrm{T}} R_2(x(t)) [u(t) + y(t)] - \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)).$$

Now, integrating over $[t_1, t_2]$ and using (5.16) yields (5.49). □

Note that with $R_2(x) \equiv I_m$ condition (5.49) is the counterpart, for discontinuous dynamical systems, of the return difference condition for continuous-time and discrete-time systems [11, 12, 52]. Next, using the extended nonlinear Kalman – Yakubovich–Popov conditions for discontinuous dynamical systems given by Theorem 4.3.1, we show that for a given nonlinear dynamical system $\mathcal{G}$ given by (5.34) and (5.35), there exists an equivalence between optimality and dissipativity. For the following result we assume that for the given discontinuous nonlinear system (5.34), if there exists a feedback control law $\phi(x)$ that minimizes the performance functional (5.36) with $R_2(x) \equiv I_m$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, then there exists a locally Lipschitz continuous, regular, and positive-definite function $V(x)$, $x \in \mathbb{R}^n$, such that (5.47) and (5.48) are satisfied. Necessary and sufficient conditions such that the aforementioned statement holds, modulo (5.48) holding, are given in Theorem 3.7.6 of [13].

Theorem 5.4.2. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.34) and (5.35). The feedback control law $u = \phi(x)$ is optimal with respect to a performance functional (5.36) with $R_2(x) \equiv I_m$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, if and only if the nonlinear system $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a locally Lipschitz continuous, regular, positive-definite, and radially unbounded storage function $V(x)$, $x \in \mathbb{R}^n$.

Proof. If the control law $\phi(x)$ is optimal with respect to a performance functional (5.36) with $R_2(x) \equiv I_m$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, then, by assumption, there exists a locally Lipschitz continuous, regular, and positive-definite function $V(x)$ such that (5.47) and (5.48) are satisfied. Hence, it follows from Lemma 5.4.1 that the solution $x(t)$, $t \geq 0$, to (5.34) satisfies

$$V(x(t_2)) \leq \int_{t_1}^{t_2} \left\{[u(t) + y(t)]^{\mathrm{T}}[u(t) + y(t)] - u^{\mathrm{T}}(t)u(t)\right\} \mathrm{d}t + V(x(t_1)), \quad 0 \leq t_1 \leq t_2,$$

which implies that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$.

Conversely, if $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a locally Lipschitz continuous, regular, and positive-definite storage function, then, with $h(x) = -\phi(x)$, $J(x) \equiv 0$, $Q = I_m$, $R = 0$, and $S = I_m$, it follows from Theorem 4.3.1 that there exists a function $\ell : \mathbb{R}^n \to \mathbb{R}^p$ such that $\phi(x) = -\frac{1}{2}\mathcal{L}_G V^{\mathrm{T}}(x)$ and, for almost all $x \in \mathbb{R}^n$,

$$0 = \min \mathcal{L}_f V(x) - \tfrac{1}{4}\mathcal{L}_G V(x)\mathcal{L}_G V^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\ell(x).$$

Now, the result follows from Theorem 5.4.1 with $L_1(x) = \ell^{\mathrm{T}}(x)\ell(x)$. □

Example 5.4.1. Consider the controlled discontinuous dynamical system $\mathcal{G}$ representing a mass sliding on a horizontal surface subject to a Coulomb frictional force given in Example 4.2.1. Let $V(x) = x^2$ and note that $\mathcal{L}_f V(x) = \{-|x|\}$ and $\mathcal{L}_G V(x) = \{2x\}$ for almost all $x \in \mathbb{R}$. Next, it follows that

$$\mathcal{L}_{\tilde{f}} V(x) = -|x| - L_2(x)x - 2x^2,$$

where $\tilde{f} \triangleq f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)$ with $R_2(x) \equiv 1$. Let $L(x,u) = L_1(x) + L_2(x)u + u^2$. Now, $L_2(x) = 2x$ satisfies $\max \mathcal{L}_{\tilde{f}} V(x) < 0$ for almost all $x \in \mathbb{R}$, $x \neq 0$, so that the inverse optimal control law is given by

$$\phi(x) = -\frac{1}{2}[2x + 2x] = -2x, \quad \text{a.e.} \quad x \in \mathbb{R}. \tag{5.52}$$

In this case, the performance functional $J(x_0, u(\cdot)) = \int_0^\infty L(x,u)\mathrm{d}t$, with

$$L_1(x) = 4x^2 + |x|, \quad \text{a.e.} \quad x \in \mathbb{R}, \tag{5.53}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}.$$

Furthermore, using the feedback control law (5.52) it follows that

$$\mathcal{L}_{\tilde{f}} V(x) = -|x| - 4x^2, \quad \text{a.e} \quad x \in \mathbb{R}.$$

Note that $\max \mathcal{L}_{\tilde{f}} V(x) \leq 0$. Now, let $\mathcal{R} \triangleq \{x \in \mathbb{R} : \frac{\mathrm{d}}{\mathrm{d}t} V(x) = 0 \in \mathcal{L}_{\tilde{f}} V(x)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t} V(x) = 0$ if and only if $x = 0$. Hence, since $\mathcal{M} = \{0\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$, it follows from Theorem 2.2.2 that $\mathrm{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (4.12). Now, since $V(x)$ is radially unbounded, the feedback control law (5.52) is globally strongly stabilizing.

Next, note that with $L_2(x) \equiv 0$ it follows from the above analysis that the optimal control law $\phi(x) = -x$ minimizes the cost functional

$$J(x_0, u(\cdot)) = \int_0^\infty [x^2(t) + |x(t)| + u^2(t)]\mathrm{d}t. \tag{5.54}$$

Now, it follows from Theorem 6.2 that the discontinuous nonlinear dynamical system $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = y^2 + 2uy$, where $y = -\phi(x) = x$. To show this, consider the storage function $V_{\mathrm{s}}(x) = V(x) = x^2$.

Next, with $J(x) \equiv 0$, $Q = 1$, $R = 0$, $S = 1$, and $\varepsilon = 0$, the extended Kalman–Yakubovich–Popov conditions given in Theorem 4.3.1 become

$$0 = \min \mathcal{L}_f V_{\rm s}(x) - h^2(x) + \ell^{\rm T}(x)\ell(x), \tag{5.55}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\rm s}(x) - h(x) + \ell^{\rm T}(x)\mathcal{W}(x), \tag{5.56}$$

$$0 = -\mathcal{W}^{\rm T}(x)\mathcal{W}(x), \tag{5.57}$$

$$\ell^{\rm T}(x)\ell(x) \geq [\max \mathcal{L}_f V_{\rm s}(x) - \min \mathcal{L}_f V_{\rm s}(x)]\,. \tag{5.58}$$

Now, with $h(x) = -\phi(x) = x$, $\mathcal{W}(x) = 0$, and $L_1(x) = \ell^{\rm T}(x)\ell(x)$, conditions (5.55)–(5.57) are satisfied. Furthermore, (5.58) is equivalent to (5.48) which is satisfied since $\phi(x) = -x$ is optimal. Hence, it follows from Theorem 4.3.1 that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^2 + 2uy$. △

Example 5.4.2. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given in Example 5.3.1. Note that with $R_2(x) \equiv 1$ and $L_2(x) \equiv 0$ it follows from the analysis given in Example 5.3.1 that the optimal control law $\phi(x) = -\frac{1}{2}\,\mathrm{sign}(x_2)$ minimizes the cost functional

$$J(x_0, u(\cdot)) = \int_0^\infty [\frac{1}{2} + \frac{1}{4}\,\mathrm{sign}^2(x_2(t)) + u^2(t)]\mathrm{d}t.$$

Now, it follows from Theorem 5.4.2 that the discontinuous nonlinear dynamical system $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^2 + 2uy$, where $y = -\phi(x) = \frac{1}{2}\,\mathrm{sign}(x_2)$. To show this, consider the storage function $V_{\rm s}(x) = V(x) = |x_1| + |x_2|$. Next, with $J(x) \equiv 0$, $Q = 1$, $R = 0$, and $S = 1$, the extended Kalman–Yakubovich–Popov conditions given in Theorem 4.3.1 become

$$0 = \min \mathcal{L}_f V_{\rm s}(x) - h^{\rm T}(x)h(x) + \ell^{\rm T}(x)\ell(x), \tag{5.59}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\rm s}(x) - h^{\rm T}(x) + \ell^{\rm T}(x)\mathcal{W}(x), \tag{5.60}$$

$$0 = -\mathcal{W}^{\rm T}(x)\mathcal{W}(x), \tag{5.61}$$

$$\ell^{\mathrm{T}}(x)\ell(x) \geq [\max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x)], \quad \text{a.e.} \quad x \in \mathbb{R}^2. \tag{5.62}$$

Next, it was shown in Example 5.3.1 that $\max \mathcal{L}_f V_{\mathrm{s}}(x) = 0$, $\min \mathcal{L}_f V_{\mathrm{s}}(x) = -\frac{1}{2}$, and $\mathcal{L}_G V_{\mathrm{s}}(x) = \{\mathrm{sign}(x_2)\}$. Now, with $h(x) = -\phi(x) = \frac{1}{2}\,\mathrm{sign}(x_2)$, $\mathcal{W}(x) = 0$, and $L_1(x) = \ell^{\mathrm{T}}(x)\ell(x)$, conditions (5.59)–(5.61) are satisfied. Furthermore, (5.62) is equivalent to (5.48) which is satisfied since $\phi(x) = -\frac{1}{2}\,\mathrm{sign}(x_2)$ is optimal. Hence, it follows from Theorem 4.3.1 that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^2 + 2uy$. △

Next, we present disk margins for the nonlinear-nonquadratic optimal regulator given by Theorem 5.4.1. First, we consider the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is a constant diagonal matrix.

Theorem 5.4.3. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.34) and (5.35) where $\phi(x)$ is a strongly stabilizing feedback control law given by (5.44) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (5.47) and (5.48) with $\theta \in \mathbb{R}$ such that $0 < \theta < 1$. If $R_2(x) \equiv \mathrm{diag}[r_1, \ldots, r_m]$, where $r_i > 0$, $i = 1, \ldots, m$, then the discontinuous nonlinear system $\mathcal{G}$ has a strong structured disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$. If, in addition, $R_2(x) \equiv I_m$, then the discontinuous nonlinear system $\mathcal{G}$ has a strong disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.

Proof. Note that for all $u(t) \in U$ and almost all $t_1, t_2 \geq 0$, $t_1 < t_2$, it follows from Lemma 5.4.1 that the solution $x(t)$, $t \geq 0$, to (5.34) satisfies

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} \left\{[u(t) + y(t)]^{\mathrm{T}} R_2 [u(t) + y(t)] - \theta^2 u^{\mathrm{T}}(t) R_2 u(t)\right\} \mathrm{d}t.$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2}V(x)$, $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = u^{\mathrm{T}} R_2 y + \frac{1-\theta^2}{2} u^{\mathrm{T}} R_2 u + \frac{1}{2} y^{\mathrm{T}} R_2 y$. Now, the result is a direct consequence of Corollary 4.4.1 and Definitions 5.2.4 and 5.2.3 with $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$. $\square$

Next, we consider the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is not a diagonal constant matrix. For the following result define

$$\bar{\gamma} \triangleq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \sigma_{\max}(R_2(x)), \quad \underline{\gamma} \triangleq \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \sigma_{\min}(R_2(x)), \tag{5.63}$$

where $R_2(x)$ is such that $\bar{\gamma} < \infty$ and $\underline{\gamma} > 0$.

Theorem 5.4.4. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.34) and (5.35) where $\phi(x)$ is a strongly stabilizing feedback control law given by (5.44) and suppose $V(x)$, $x \in \mathbb{R}^n$, satisfies (5.47) and (5.48) with $\theta \in \mathbb{R}$ such that $0 < \theta < 1$. Then the discontinuous nonlinear system $\mathcal{G}$ has a strong disk margin $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where $\eta \triangleq \sqrt{\underline{\gamma}/\bar{\gamma}}$.

Proof. Note that for almost all $u(t) \in U$ and $t_1, t_2 \geq 0$, $t_1 < t_2$, it follows from Lemma 5.4.1 that the solution $x(t)$, $t \geq 0$, to (5.34) satisfies

$$\begin{aligned} V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} & \big\{[u(t) + y(t)]^{\mathrm{T}} R_2(x(t))[u(t) + y(t)] \\ & -\theta^2 u^{\mathrm{T}}(t) R_2(x(t)) u(t)\big\}\mathrm{d}t, \end{aligned}$$

which implies that

$$\begin{aligned} V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} & \big\{\bar{\gamma}[u(t) + y(t)]^{\mathrm{T}}[u(t) + y(t)] \\ & -\underline{\gamma}\theta^2 u^{\mathrm{T}}(t) u(t)\big\}\mathrm{d}t. \end{aligned}$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2\underline{\gamma}} V(x)$, $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = u^{\mathrm{T}} y + \frac{1-\eta^2\theta^2}{2} u^{\mathrm{T}} u + \frac{1}{2} y^{\mathrm{T}} y$. Now, the result is a direct consequence of Corollary 4.4.1 and Definition 5.2.3 with $\alpha = \frac{1}{1+\eta\theta}$ and $\beta = \frac{1}{1-\eta\theta}$.

$\square$

Next, using Theorem 2.2.2 we provide an alternative result that guarantees sector and gain margins for the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal.

Theorem 5.4.5. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (5.34) and (5.35) where $\phi(x)$ is a strongly stabilizing feedback control law given by (5.44) and suppose $V(x)$, $x \in \mathbb{R}^n$, satisfies (5.47) and (5.48) with $\theta \in \mathbb{R}$ such that $0 < \theta < 1$. Furthermore, let $R_2(x) = \operatorname{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$. If $\mathcal{G}$ is strongly zero-state observable, then the discontinuous nonlinear system $\mathcal{G}$ has a strong sector (and, hence, gain) margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.

Proof. Let $\Delta(-y) = \sigma(-y)$, where $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is a static nonlinearity such that $\sigma(0) = 0$, $\sigma(v) = [\sigma_1(v_1), \ldots, \sigma_m(v_m)]^{\mathrm{T}}$, and $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, for all $v_i \neq 0$, $i = 1, \ldots, m$, where $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$; or, equivalently, $(\sigma_i(v_i) - \alpha v_i)(\sigma_i(v_i) - \beta v_i) < 0$, for all $v_i \neq 0$, $i = 1, \ldots, m$. In this case, the closed-loop discontinuous system (5.34) and (5.35) with $u = \sigma(-y)$ is given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\sigma(\phi(x(t))), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0. \tag{5.64}$$

Next, consider the locally Lipschitz continuous and regular Lyapunov function candidate $V(x)$, $x \in \mathbb{R}^n$. Now, it follows from (5.47), (5.48), and (5.51) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x) &\leq \max \mathcal{L}_{f+G\sigma}V(x) \\
&\leq \max\left[\mathcal{L}_f V(x) + \mathcal{L}_{G\sigma}V(x)\right] \\
&= \max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\sigma(\phi(x)) \\
&\leq \min \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\sigma(\phi(x)) + L_1(x) - \tfrac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x) \\
&\quad + (1-\theta^2)\left[\sigma(\phi(x)) + \tfrac{1}{2(1-\theta^2)} R_2^{-1}(x) L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x) \\
&\quad \times \left[\sigma(\phi(x)) + \tfrac{1}{2(1-\theta^2)} R_2^{-1}(x) L_2^{\mathrm{T}}(x)\right] \\
&= \min \mathcal{L}_f V(x) + L_1(x) + \mathcal{L}_G V(x)\sigma(\phi(x)) + (1-\theta^2)\sigma^{\mathrm{T}}(\phi(x)) R_2(x) \sigma(\phi(x))
\end{aligned}$$

$$
\begin{aligned}
&+L_2(x)\sigma(\phi(x)) \\
&= \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - 2\phi^{\mathrm{T}}(x)R_2(x)\sigma(\phi(x)) + (1-\theta^2)\sigma^{\mathrm{T}}(\phi(x))R_2(x)\sigma(\phi(x)) \\
&= \sum_{i=1}^{m} r_i(x)(\tfrac{1}{\beta}\sigma_i(-y_i) + y_i)(\tfrac{1}{\alpha}\sigma_i(-y_i) + y_i) \\
&= \tfrac{1}{\alpha\beta}\sum_{i=1}^{m} r_i(x)\left(\sigma_i(-y_i) + \alpha y_i\right)\left(\sigma_i(-y_i) + \beta y_i\right) \\
&\leq 0, \quad \text{a.e.} \quad x \in \mathbb{R}^n,
\end{aligned}
$$

which implies that the closed-loop discontinuous system (5.64) is strongly Lyapunov stable.

Next, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^n : \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0 \in \mathcal{L}_{f+G\sigma}V(x)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$ if and only if $y = 0$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that $\mathcal{M} \triangleq \{x \in \mathbb{R}^n : \ x = 0\}$ is the largest weakly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 2.2.2 that $x(t) \to \mathcal{M} = \{0\}$ as $t \to \infty$. Thus, the closed-loop discontinuous system (5.64) is globally strongly asymptotically stable for all $\sigma(\cdot)$ such that $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, $v_i \neq 0$, $i = 1, \ldots, m$, which implies that the discontinuous nonlinear system $\mathcal{G}$ given by (5.34) and (5.35) has strong sector (and, hence, gain) margin (α, β). □

Note that in the case where $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal, Theorem 5.4.5 guarantees larger strong gain and sector margins to the strong gain and sector margin guarantees provided by Theorem 5.4.4. However, Theorem 5.4.5 does not provide strong disk margin guarantees.

Chapter 6

On Almost Consensus of Multiagent Systems with Inaccurate Sensor Measurements

6.1. Introduction

In this chapter, we consider a multiagent consensus problem in which agents possess sensors with limited accuracy. Specifically, we develop consensus control protocols for continuous- and discrete-time network systems that guarantee that the agents reach an almost consensus state and converge to a set centered at the centroid of the agents' initial locations. This set is shown to be time-varying, in the sense that only the differences between agent positions are, in the limit, small.

For discrete-time network systems, we also use difference inclusions and set-valued analysis to describe the inaccurate sensor measurement problem formulation. Set-valued analysis has been previously used for consensus control. In [49], the author uses set-valued Lyapunov functions to study convergence of multiagent dynamical systems. The approach involves constructing set-valued Lyapunov functions from convex sets that depend on the agent states. In [2, 46, 49], the authors address stability of each equilibrium point in the sense that the system solutions approach an equilibrium from a neighborhood of equilibria. Reference [46] considers barycentric coordinate maps, whereas [49] and [2] consider difference equations and difference inclusions, respectively.

Necessary and sufficient conditions for semistability for multiagent consensus problems using set-valued Lyapunov analysis are presented in [28]. More recently, the authors in [76] consider an asynchronous rendezvous problem using set-valued consensus theory. Specifically, a design strategy for multiagent consensus is developed by requiring two consecutive way-points to be included within a minimum convex region covering the two associated anticipated-way-point sets.

The proposed set-valued consensus protocol builds on the framework of [49], [28], and [29] to develop almost consensus protocols for multiagent systems with uncertain interagent measurements. Specifically, the proposed protocol algorithm modifies the set-valued consensus update maps of the agents by assuming that the locations of all agents, including the agents calculating the update map, are within a ball of radius r. However, since the update sets of our design protocol do not satisfy a strict convexity assumption, our results go beyond the results of [49] by employing a set-valued invariance principle.

6.2. Consensus Control Problem with Uncertain Interagent Location Measurements

In this chapter, we consider a multiagent network in which N agents reach an almost consensus state and we use the terminology *agent state* and *agent location* interchangeably. Each agent $i \in \{1, \ldots, N\}$ has a sensor with accuracy r, that is, each agent i can detect the location of the other agents with an accuracy of up to a ball of radius r centered at the actual location of the other agents. Specifically, the approximate location of agent i as measured by agent j is given by the set

$$\mathcal{X}_i = \{p \in \mathbb{R}^n : \|p - x_i\|_2 \leq r\}, \quad i = 1, \ldots, N.$$

The network consensus problem considered in this chapter involves the design of a dynamic protocol that guarantees almost system state equipartition, that is, the difference between any two agent states decreases to below a certain threshold that is dependent on the sensor accuracy r. Specifically, each agent i uses an update protocol resulting in a closed loop system similar to (2.12) or (2.13). However, since only approximate information of the location of the other agents is available at any given instant of time, the update protocol is constructed using approximate location information only. In particular, for a discrete-time network system the update protocol for a connected graph has the form

$$x_i(k+1) \in \mathcal{F}_i(x(k)) \triangleq x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_{\text{in}}(i)} (\mathcal{X}_j(k) - x_i(k)), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \tag{6.1}$$

where, $i = 1, \ldots, N$, $x \triangleq [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, and $\mathcal{X}_j - x_i$ denotes the set of all vectors $z \in \mathbb{R}^n$ such that $z = y - x_i$ with $y \in \mathcal{X}_j$. Note that for the protocol given by (2.13) every agent has information of the exact location of the other agents, whereas for the protocol given by (6.1) only approximate location information of the other agents is available.

To further elucidate the protocol architecture given by (6.1), consider a connected network consisting of three agents. In this case, the update protocol for Agent 1 is given by

$$x_1(k+1) \in \mathcal{F}_1(x(k)) = x_1(k) + \varepsilon(\mathcal{X}_1(k) - x_1(k) + \mathcal{X}_2(k) - x_1(k) + \mathcal{X}_3(k) - x_1(k)),$$
$$x_1(0) = x_{10}, \quad k \in \overline{\mathbb{Z}}_+,$$

where the sets $\mathcal{X}_2 - x_1$ and $\mathcal{X}_3 - x_1$ are depicted in Figure 6.1.

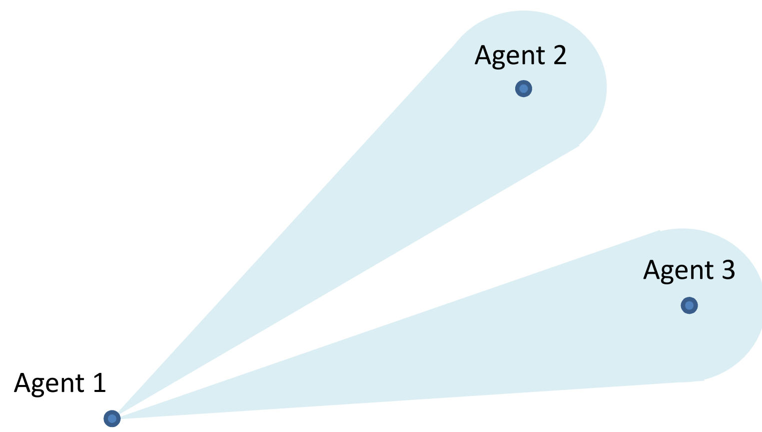

Figure 6.1. Visualization of sets $\mathcal{X}_2 - x_1$ and $\mathcal{X}_3 - x_1$ used in agent's 1 update map.

6.3. Continuous-Time Consensus with a Connected Graph Topology

In this section, we consider the continuous-time consensus problem over an undirected network with a connected graph topology. We assume that only approximate information of the location of neighboring agents is available at any given instant of time with ith agent uncertainty satisfying $\|d_i(t)\|_2 \leq r$, $t \geq 0$, for $i = 1, \ldots, N$. In particular, we consider the update protocol for agent i given by

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}(i)} (z_j(t) - z_i(t)), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N, \tag{6.2}$$

where

$$z_j(t) - z_i(t) \triangleq (x_j(t) - d_j(t)) - (x_i(t) - d_i(t)).$$

In this case, it follows from (6.2) that

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}(i)} (x_j(t) - x_i(t)) + \sum_{j \in \mathcal{N}(i)} (d_i(t) - d_j(t)), \quad x_i(0) = x_{i0}, \quad t \geq 0,$$
$$i = 1, \ldots, N,$$

or, equivalently, in compact form

$$\dot{x}(t) = -\tilde{\mathcal{L}}x(t) + \tilde{\mathcal{L}}d(t), \quad x(0) = x_0, \quad t \geq 0, \tag{6.3}$$

where $\tilde{\mathcal{L}} \triangleq I_n \otimes \mathcal{L} \in \mathbb{R}^{nN \times nN}$, $\mathcal{L} \in \mathbb{R}^{N \times N}$ denotes the graph Laplacian, $\otimes$ denotes Kronecker product, $x \triangleq [x_1^1, \ldots, x_N^1, \ldots, x_1^n, \ldots, x_N^n]^{\mathrm{T}}$, $d \triangleq [d_1^1, \ldots, d_N^1, \ldots, d_1^n, \ldots, d_N^n]^{\mathrm{T}}$, and x_i^j and d_i^j denote the jth component of x_i and d_i, respectively.

Although our results can be directly extended to the case of (6.3), for simplicity of exposition, we will focus on individual agent states evolving in $\mathbb{R}$ (i.e., $n = 1$). In this case, (6.3) becomes

$$\dot{x}(t) = -\mathcal{L}x(t) + \mathcal{L}d(t), \quad x(0) = x_0, \quad t \geq 0. \tag{6.4}$$

For the statement of the next result let $\mathbf{e}_N \triangleq [1, \ldots, 1]^{\mathrm{T}}$ denote the ones vector of order N and $\overline{x} \triangleq \frac{1}{N}\mathbf{e}_N^{\mathrm{T}} x$. Furthermore, recall that the Laplacian of an undirected connected graph is a symmetric positive semidefinite matrix with a single zero eigenvalue [48]; specifically, the eigenvalues of the graph Laplacian are given by $0 = \lambda_{\min}(\mathcal{L}) \triangleq \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \lambda_3(\mathcal{L}) \leq \cdots \leq \lambda_N(\mathcal{L}) \triangleq \lambda_{\max}(\mathcal{L})$. Hence, the Schur decomposition of $-\mathcal{L}$ is given by $-\mathcal{L} = P_\Sigma \Sigma P_\Sigma^{\mathrm{T}}$, where $P_\Sigma \triangleq [p_1, \ldots, p_{N-1}, \frac{1}{\sqrt{N}}\mathbf{e}_N]$, with $p_i \in \mathbb{R}^N$, $i = 1, \ldots, N-1$,

$$\Sigma \triangleq \begin{bmatrix} \Sigma_0 & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix},$$

and $\Sigma_0 \in \mathbb{R}^{(N-1)\times(N-1)}$ is Hurwitz.

Theorem 6.3.1. Consider an undirected network of N agents with a connected graph topology given by (6.4). Then, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq \frac{\lambda_N(\mathcal{L})\sqrt{N}r}{\lambda_2(\mathcal{L})}$.

Proof. First, define $\delta(t) \triangleq x(t) - \mathbf{e}_N \overline{x}(t)$ and note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{N}\mathbf{e}_N^{\mathrm{T}} x(t)\right) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}}(-\mathcal{L}x(t) + \mathcal{L}d(t)) = 0_N,$$

where we used the fact that $\mathcal{L}\mathbf{e}_N = 0_N$ and $\mathcal{L} = \mathcal{L}^{\mathrm{T}}$. Hence, $\overline{x}(t) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}}x(t) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}}x(0)$, $t \geq 0$, which shows that the centroid of the network does not change over time in the presence of time-varying interagent measurement uncertainties. Next, differentiating $\delta(t)$ with respect to time yields

$$\begin{aligned}\dot{\delta}(t) &= \dot{x}(t) - \mathbf{e}_N\dot{\overline{x}}(t) \\ &= -\mathcal{L}x(t) + \mathcal{L}d(t) \\ &= -\mathcal{L}\left[\delta(t) + \mathbf{e}_N\overline{x}(t)\right] + \mathcal{L}d(t) \\ &= -\mathcal{L}\delta(t) + \mathcal{L}d(t), \quad \delta(0) = \delta_0, \quad t \geq 0. \end{aligned} \tag{6.5}$$

Introducing the transformation $q(t) \triangleq P_\Sigma^{\mathrm{T}}\delta(t)$, it follows from (6.5) that

$$\begin{aligned}\dot{q}(t) &= P_\Sigma^{\mathrm{T}}\dot{\delta}(t) \\ &= -P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma P_\Sigma^{\mathrm{T}}\delta(t) + P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma P_\Sigma^{\mathrm{T}}d(t) \\ &= -P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma q(t) + P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma\overline{d}(t), \quad q(0) = q_0, \quad t \geq 0,\end{aligned}$$

where $\overline{d}(t) \triangleq P_\Sigma^{\mathrm{T}}d(t)$, and hence,

$$\dot{q}(t) = \begin{bmatrix} \Sigma_0 & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix}\left[q(t) - \overline{d}(t)\right], \quad q(0) = q_0, \quad t \geq 0. \tag{6.6}$$

Now, it follows from (6.6) that

$$\dot{q}_1(t) = \Sigma_0 q_1(t) - \Sigma_0\overline{d}_1(t), \quad q_1(0) = q_{10}, \quad t \geq 0, \tag{6.7}$$

$$\dot{q}_2(t) = 0, \quad q_2(0) = q_{20}, \tag{6.8}$$

where

$$q_1(t) \triangleq \left[\begin{array}{cc} I_{(N-1)\times(N-1)} & 0_{(N-1)\times 1}\end{array}\right] q(t), \quad \overline{d}_1(t) \triangleq \left[\begin{array}{cc} I_{(N-1)\times(N-1)} & 0_{(N-1)\times 1}\end{array}\right] \overline{d}(t),$$

and $q_2 \in \mathbb{R}$. Furthermore, note that $q_{20} = 0$ since $\mathbf{e}_N^{\mathrm{T}}\delta(t) = \mathbf{e}_N^{\mathrm{T}}x(t) - \frac{1}{N}\mathbf{e}_N^{\mathrm{T}}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}x(t) = 0$.

Next, consider the Lyapunov function candidate $V : \mathbb{R}^{(N-1)} \to \mathbb{R}$ given by $V(q_1) = q_1^{\mathrm{T}} S q_1$, where $S = S^{\mathrm{T}} > 0$, $S \in \mathbb{R}^{(N-1)\times(N-1)}$, satisfies

$$0 = \Sigma_0^{\mathrm{T}} S + S\Sigma_0 + Q, \tag{6.9}$$

with $Q = Q^{\mathrm{T}} > 0$ and $Q \in \mathbb{R}^{(N-1)\times(N-1)}$. Now, note that the derivative of $V(q_1)$ along the trajectories of (6.7) is given by

$$\begin{aligned}
\dot{V}(q_1(t)) &= -q_1^{\mathrm{T}}(t)Qq_1(t) - 2q_1^{\mathrm{T}}(t)S\Sigma_0\overline{d}_1(t) \\
&\le -\lambda_{\min}(Q)\|q_1(t)\|_2^2 + 2\sigma_{\max}(S\Sigma_0)\sigma_{\max}\left(\left[\begin{array}{cc} I_{(N-1)\times(N-1)} & 0_{(N-1)\times 1} \end{array}\right]\right) \\
&\quad \times\sigma_{\max}(P_\Sigma^{\mathrm{T}})\|d(t)\|_2\|q_1(t)\|_2 \\
&\le -\lambda_{\min}(Q)\|q_1(t)\|_2^2 + 2\sigma_{\max}(S\Sigma_0)\sqrt{N}r\|q_1(t)\|_2 \\
&= -\|q_1(t)\|_2\left[\lambda_{\min}(Q)\|q_1(t)\|_2 - 2\sigma_{\max}(S\Sigma_0)\sqrt{N}r\right], \quad t \ge 0,
\end{aligned} \tag{6.10}$$

where we used the fact that $\sigma_{\max}\left(\left[\begin{array}{cc} I_{(N-1)\times(N-1)} & 0_{(N-1)\times 1} \end{array}\right]\right) = 1$, $\sigma_{\max}(P_\Sigma^{\mathrm{T}}) = 1$, and $\|d(t)\|_2 \le \sqrt{N}r$, $t \ge 0$.

Next, it follows from (6.10) that $\dot{V}(q_1(t)) \le 0$ for $\|q_1(t)\|_2 \ge \frac{2\sigma_{\max}(S\Sigma_0)\sqrt{N}r}{\lambda_{\min}(Q)} \triangleq \beta$ and $t \ge 0$, and hence, $q_1(t)$, $t \ge 0$, is decreasing for $\|q_1(t)\|_2 > \beta$. Moreover, since $\dot{q}_2(t) = 0$, $t \ge 0$, and $q_2(0) = 0$, $q_2(t) = 0$ for all $t \ge 0$. Hence, it follows from the definition of $q(t)$ and (6.10) that

$$\|\delta(t)\|_2 = \left\|\left[\begin{array}{c} q_1(t) \\ q_2(t) \end{array}\right]\right\|_2 = \|q_1(t)\|_2 \le \beta$$

as $t \to \infty$. Now, setting $Q = -\Sigma_0$ it follows from (6.9) that $S = \frac{1}{2}I_{(N-1)}$, and hence, $\|q_1(t)\|_2 = \|x(t) - \mathbf{e}_N\overline{x}\|_2 \le \beta$, $t \ge 0$, where

$$\beta = \frac{2\sigma_{\max}(\frac{1}{2}\Sigma_0)\sqrt{N}r}{\lambda_{\min}(-\Sigma_0)} = \frac{\lambda_N(\mathcal{L})\sqrt{N}r}{\lambda_2(\mathcal{L})}, \tag{6.11}$$

which completes the proof. □

Next, we apply Theorem 6.3.1 to an all-to-all connected graph network. Note that in this case, $\mathcal{L} = NI_N - E_N$, where $E_N \triangleq \mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}$ denotes the *ones matrix* of order

$N \times N$. Since $\operatorname{rank} E_N = 1$, E_N has only one nonzero eigenvalue equal to N with corresponding eigenvector $\mathbf{e}_N$. Next, note that

$$\det[\lambda I_N - \mathcal{L}] = \det[\lambda I_N - (N I_N - E_N)] = \det[(\lambda - N) I_N + E_N].$$

Hence, the eigenvalues of $\mathcal{L}$ are the eigenvalues of $-E_N$ shifted by N, that is, $\operatorname{spec}(-E_N) = \{0, N, \ldots, N\}$. Now, with $\lambda_2(\mathcal{L}) = \cdots = \lambda_N(\mathcal{L}) = N$, it follows from Theorem 6.3.1 that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}\|_2 \le \sqrt{N} r$.

Alternatively, we can arrive at the same result directly by considering the update protocol for the ith agent given by

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j=1}^{N}[(x_j(t) - d_j(t)) - (x_i(t) - d_i(t))] = \overline{x}(t) - x_i(t) - \overline{d}(t) + d_i(t),$$
$$x_i(0) = x_{i0}, \quad t \ge 0, \quad i = 1, \ldots, N, \qquad (6.12)$$

where $\overline{x}(t) \triangleq \frac{1}{N}\sum_{j=1}^{N} x_j(t) \equiv \overline{x}$ and $\overline{d}(t) \triangleq \frac{1}{N}\sum_{j=1}^{N} d_j(t)$. First, note that it can be shown that $\limsup_{t\to\infty} \|x_i(t) - x_j(t)\|_2 \le 2r$ for every $i, j = 1, \ldots, N$. To see this, for $i, j = 1, \ldots, N$, it follows from (6.12) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&\left(\frac{1}{2}\|x_i(t) - x_j(t)\|_2^2\right) \\
&= (x_i(t) - x_j(t))^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}(x_i(t) - x_j(t)) \\
&= (x_i(t) - x_j(t))^{\mathrm{T}}[\overline{x} - x_i(t) - \overline{d}(t) + d_i(t) - (\overline{x} - x_j(t) - \overline{d}(t) + d_j(t))] \\
&= -\|x_i(t) - x_j(t)\|_2^2 + (x_i(t) - x_j(t))^{\mathrm{T}}(d_i(t) - d_j(t)) \\
&\le -\|x_i(t) - x_j(t)\|_2^2 + 2r\|x_i(t) - x_j(t)\|_2, \\
&\qquad\qquad x_i(0) - x_j(0) = x_{i0} - x_{j0}, \quad t \ge 0,
\end{aligned}$$

where the last inequality follows from the fact that

$$\|d_i(t) - d_j(t)\|_2 \le \|d_i(t)\|_2 + \|d_j(t)\|_2 \le 2r, \quad t \ge 0.$$

Hence, $\|x_i(t) - x_j(t)\|_2$ is a decreasing function of time as long as $\|x_i(t) - x_j(t)\|_2 > 2r$, $t \ge 0$. Now, it follows that $\|x_i(t) - x_j(t)\|_2 \le 2r$ as $t \to \infty$ for all $i, j = 1, \ldots, N$.

Next, since $\overline{x}(t) \equiv \overline{x}$, it follows that $\|x_i(t)-\overline{x}\|_2 \leq r$ as $t \to \infty$ for all $i = 1, \ldots, N$. Furthermore, since

$$\|x(t) - \mathbf{e}_N \overline{x}\|_2^2 = \sum_{i=1}^{N} \|x_i(t) - \overline{x}\|_2^2 \leq N r^2$$

as $t \to \infty$, it follows that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}\|_2 \leq \sqrt{N} r$, which is identical to the result obtained by applying Theorem 6.3.1.

6.4. Discrete-Time Consensus with a Connected Graph Topology

In this section, we consider the discrete-time consensus problem over an undirected network with a connected graph topology. Once again, we assume that only approximate information of the location of neighboring agents is available at any given instant of time with ith agent uncertainty satisfying $\|d_i(k)\|_2 \leq r$, $k \in \overline{\mathbb{Z}}_+$, for $i = 1, \ldots, N$. In particular, we consider the update protocol for agent i given by

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}(i)} (z_j(k) - z_i(k)), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, N, \tag{6.13}$$

where

$$z_j(k) - z_i(k) \triangleq (x_j(k) - d_j(k)) - (x_i(k) - d_i(k))$$

and $\varepsilon > 0$. In this case, it follows from (6.13) that

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}(i)} (x_j(k) - x_i(k)) + \varepsilon \sum_{j \in \mathcal{N}(i)} (d_i(k) - d_j(k)), \quad x_i(0) = x_{i0},$$
$$k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, N,$$

or, equivalently, in compact form

$$x(k+1) = \tilde{\mathcal{P}} x(k) + \varepsilon \tilde{\mathcal{L}} d(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{6.14}$$

where $\tilde{\mathcal{L}} \triangleq I_n \otimes \mathcal{L} \in \mathbb{R}^{nN \times nN}$, $\tilde{\mathcal{P}} \triangleq I_n \otimes \mathcal{P} \in \mathbb{R}^{nN \times nN}$, $\mathcal{L} \in \mathbb{R}^{N \times N}$ denotes the graph Laplacian, $\mathcal{P} \triangleq I_N - \varepsilon \mathcal{L} \in \mathbb{R}^{N \times N}$ denotes the Perron matrix, $x \triangleq$

$[x_1^1, \ldots, x_N^1, \ldots, x_1^n, \ldots, x_N^n]^{\mathrm{T}}$, $d \triangleq [d_1^1, \ldots, d_N^1, \ldots, d_1^n, \ldots, d_N^n]^{\mathrm{T}}$, and x_i^j and d_i^j denote the jth component of x_i and d_i, respectively.

Although our results can be directly extended to the case of (6.14), for simplicity of exposition, we will focus on individual agent states evolving in $\mathbb{R}$ (i.e., $n = 1$). In this case, (6.14) becomes

$$x(k+1) = \mathcal{P}x(k) + \varepsilon \mathcal{L} d(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+. \tag{6.15}$$

For the statement of the next result define $\Delta_{\max} \triangleq \max_{i \in \{1,\ldots,N\}} \deg(i)$.

Theorem 6.4.1. Consider an undirected network of N agents with a connected graph topology given by (6.15) and let $\varepsilon \in \left(0, \frac{1}{\Delta_{\max}}\right)$. Then,

$$\limsup_{k\to\infty} \|x(k) - \mathbf{e}_N \overline{x}(k)\|_2 \leq \frac{\varepsilon \lambda_{\max}(\mathcal{L})\sqrt{N} r}{1 - \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N \mathbf{e}_N^{\mathrm{T}}\right)}.$$

Proof. First, define $\delta(k) \triangleq x(k) - \mathbf{e}_N \overline{x}(k)$ and note that $\overline{x}(k+1) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}} x(k+1) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}}(x(k) + \varepsilon(-\mathcal{L}x(k) + \mathcal{L}d(k)) = \overline{x}(k)$, where we used the fact that $\mathcal{L}\mathbf{e}_N = 0_N$ and $\mathcal{L} = \mathcal{L}^{\mathrm{T}}$. Hence, $\overline{x}(k) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}} x(k) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}} x(0)$, $k \in \overline{\mathbb{Z}}_+$, which shows that the centroid of the network does not change over time in the presence of time-varying interagent measurement uncertainties. Next, evaluating $\delta(k+1)$, $k \in \overline{\mathbb{Z}}_+$, yields

$$\begin{aligned}
\delta(k+1) &= x(k+1) - \mathbf{e}_N \overline{x}(k+1) \\
&= \mathcal{P}x(k) + \varepsilon \mathcal{L} d(k) - \frac{1}{N}\mathbf{e}_N \mathbf{e}_N^{\mathrm{T}} \left[\mathcal{P}x(k) + \varepsilon \mathcal{L} d(k)\right] \\
&= \mathcal{P}\left[x(k) - \frac{1}{N}\mathbf{e}_N \mathbf{e}_N^{\mathrm{T}} x(k)\right] + \left[I - \frac{1}{N}\mathbf{e}_N \mathbf{e}_N^{\mathrm{T}}\right] \varepsilon \mathcal{L} d(k) \\
&= \left[\mathcal{P} - \frac{1}{N}\mathbf{e}_N \mathbf{e}_N^{\mathrm{T}}\right] \delta(k) + \varepsilon \mathcal{L} d(k), \quad \delta(0) = \delta_0, \quad k \in \overline{\mathbb{Z}}_+.
\end{aligned} \tag{6.16}$$

Now, considering a Lyapunov function candidate $V : \mathbb{R}^{(N-1)} \to \mathbb{R}$ given by $V(\delta) = \|\delta\|_2$ and recalling that $\rho(M) = \|M\|_2$ for an arbitrary symmetric matrix M, it follows from (6.16) that

$$\begin{aligned} V(\delta(k+1)) &= \|\delta(k+1)\|_2 \\ &\leq \left\|(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}})\delta(k)\right\|_2 + \|\varepsilon\mathcal{L}d(k)\|_2 \\ &\leq \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)\|\delta(k)\|_2 + \varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r \\ &= \left(\rho(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}) + \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{\|\delta(k)\|_2}\right)V(\delta(k)), \quad k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{6.17}$$

Hence, it follows from (6.17) that $V(\delta(k+1)) < V(\delta(k))$ for $\rho(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}) + \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{\|\delta(k)\|_2} < 1$ and $k \in \overline{\mathbb{Z}}_+$. Now, recalling that all the eigenvalues of the Perron matrix of an undirected connected graph with $\varepsilon \in \left(0, \frac{1}{\Delta_{\max}}\right)$ are located in the unit circle and only one eigenvalue has an absolute value of 1 [55], it follows that $\rho(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}) < 1$. Hence, it follows from (6.17) that

$$\|\delta(k)\|_2 \leq \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{1 - \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)}$$

as $k \to \infty$, which completes the proof. □

Remark 6.4.1. Note that

$$\begin{aligned} \det\left[\lambda I_N - \left(\mathcal{P} - \frac{1}{N}E_N\right)\right] &= \det\left[\lambda I_N - \left(I_N - \varepsilon\mathcal{L} - \frac{1}{N}E_N\right)\right] \\ &= \det\left[(\lambda - 1)\, I_N - \left(-\varepsilon\mathcal{L} - \frac{1}{N}E_N\right)\right]. \end{aligned} \tag{6.18}$$

Now, since E_N has only one nonzero eigenvalue equal to N with the corresponding eigenvector $\mathbf{e}_N$ and $\mathcal{L}$ has only one zero eigenvalue with the corresponding eigenvector $\mathbf{e}_N$, it follows that $\mathrm{spec}(-\varepsilon\mathcal{L} - \frac{1}{N}E_N) = \{-1, -\varepsilon\lambda_2(\mathcal{L}), \ldots, -\varepsilon\lambda_N(\mathcal{L})\}$. Thus, it follows from (6.18) that $\mathrm{spec}(\mathcal{P} - \frac{1}{N}E_N) = \{0, (1 - \varepsilon\lambda_2(\mathcal{L})), \ldots, (1 - \varepsilon\lambda_N(\mathcal{L}))\}$. Hence, $\rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right) = \max\{|(1 - \varepsilon\lambda_2(\mathcal{L}))|, |(1 - \varepsilon\lambda_N(\mathcal{L}))|\}$.

Next, we apply Theorem 6.4.1 to an all-to-all connected graph network. Note that in this case, $\mathcal{L} = NI_N - E_N$. Next, recall that $\lambda_2(\mathcal{L}) = \cdots = \lambda_N(\mathcal{L}) = N$, and hence, for $\varepsilon \in (0, \frac{1}{N})$, it follows from Theorem 6.4.1 and Remark 6.4.1 that

$$\limsup_{k\to\infty} \|x(k) - \mathbf{e}_N\overline{x}(k)\|_2 \leq \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{1-\rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)} = \frac{\varepsilon N\sqrt{N}r}{1-(1-\varepsilon N)} = \sqrt{N}r.$$

Alternatively, we can arrive at the same result directly by considering the update protocol for the ith agent given by

$$x_i(k+1) \in \alpha\frac{1}{N}\sum_{j=1}^{N}\mathcal{X}_j(k) + (1-\alpha)x_i(k) = \mathcal{B}_{\alpha r}(\alpha\overline{x}(k)) + (1-\alpha)x_i(k),$$

$$x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \quad i = 1,\ldots,N, \quad (6.19)$$

where $\alpha \in (0,1]$ and $\overline{x}(k) \triangleq \frac{1}{N}\sum_{i=1}^{N} x_i(k) \equiv \overline{x}$. First, note that it can be shown that $\limsup_{k\to\infty}$ $\|x_i(k) - x_j(k)\|_2 \leq 2r$ for every $i,j = 1,\ldots,N$. To see this, for $i,j = 1,\ldots,N$, it follows from (6.19) that

$$x_i(k+1) - x_j(k+1) \in \mathcal{B}_{\alpha r}(\alpha x_{\mathrm{ave}}(k)) - \mathcal{B}_{\alpha r}(\alpha x_{\mathrm{ave}}(k)) + (1-\alpha)(x_i(k) - x_j(k)),$$

$$k \in \overline{\mathbb{Z}}_+, \quad (6.20)$$

which implies

$$\|x_i(k+1) - x_j(k+1)\|_2 \leq (1-\alpha)\|x_i(k) - x_j(k)\|_2 + 2r\alpha. \quad (6.21)$$

Hence, since $\|x_i(k+1) - x_j(k+1)\|_2 \leq \|x_i(k) - x_j(k)\|_2$ for $\|x_i(k) - x_j(k)\|_2 \geq 2r$, it follows that $\|x_i(k) - x_j(k)\|_2 \leq 2r$ as $k \to \infty$ for all $i,j = 1,\ldots,N$. Now, using identical arguments as in Section 6.3, it follows that $\limsup_{k\to\infty} \|x(k) - \mathbf{e}_N\overline{x}(k)\|_2 \leq \sqrt{N}r$, which is identical to the result obtained by using Theorem 6.4.1.

6.5. A Set-Valued Analysis Approach to Discrete-Time Consensus

In this section, we present a set-valued approach for the discrete-time consensus protocol considered in Section 6.4. However, before presenting the main results of this section we require some additional notation and definitions. Specifically, consider the difference inclusion

$$x(k+1) \in \mathcal{F}(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{6.22}$$

where, for every $k \in \overline{\mathbb{Z}}_+$, $x(k) \in \mathbb{R}^n$, $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a *set-valued map* that assigns sets to points, and $2^{\mathbb{R}^n}$ denotes the collection of all subsets of $\mathbb{R}^n$. The set-valued map $\mathcal{F}$ has a *nonempty value at* x if $\mathcal{F}(x) \neq \varnothing$. It is assumed that $\mathcal{F}$ has nonempty values for ever $x \in \mathbb{R}^n$. Hence, maximal solutions to (6.22) are complete, and consequently, by a *solution* of (6.22) with initial condition $x(0) = x_0$ we mean a function $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$ that satisfies (6.22).

The set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is *outer semicontinuous at* x if, for every sequence $\{x_i\}_{i=0}^{\infty}$ such that $\lim_{i\to\infty} x_i = x$, every convergent sequence $\{y_i\}_{i=0}^{\infty}$ with $y_i \in \mathcal{F}(x_i)$ satisfies $\lim_{i\to\infty} y_i \in \mathcal{F}(x)$. $\mathcal{F}$ is *continuous at* x if $\mathcal{F}$ is outer semicontinuous at x and, for every $y \in \mathcal{F}(x)$ and every convergent sequence $\{x_i\}_{i=0}^{\infty}$, there exists $y_i \in \mathcal{F}(x_i)$ such that $\lim_{i\to\infty} y_i = y$. $\mathcal{F}(x)$ is *locally bounded at* x if there exists a neighborhood $\mathcal{N}$ of x such that $\mathcal{F}(\mathcal{N}) = \cup_{z\in\mathcal{N}}\mathcal{F}(z)$ is bounded. If $\mathcal{F}$ has compact values and is locally bounded at x, then $\mathcal{F}$ is *upper semicontinuous at* x, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \mathbb{R}^n$ satisfying $\|z - x\| < \delta$, $\mathcal{F}(z) \subseteq \mathcal{F}(x) + \overline{\mathcal{B}}_\varepsilon(0)$, where $\overline{\mathcal{B}}_\varepsilon(0)$ denotes the closure of $\mathcal{B}_\varepsilon(0)$.

Given the function $\gamma : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$, the *positive limit set of* γ is the set $\Omega(\gamma)$ of points $y \in \mathbb{R}^n$ for which there exists an increasing divergent sequence $\{k_n\}_{n=0}^{\infty}$ satisfying $\lim_{n\to\infty} \gamma(k_n) = y$. We denote the positive limit set of a solution $\psi(\cdot)$ of (6.22) by $\Omega(\psi)$. The positive limit set of a bounded solution of (6.22) is nonempty, compact, and weakly forward invariant with respect to (6.22) [62].

The following theorem gives a general set-valued invariance principle using the set-valued analysis tools developed in [28].

Theorem 6.5.1. Consider the difference inclusion given by (6.22). Assume that $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is outer semicontinuous and locally bounded with nonempty values for all $x \in \mathbb{R}^n$. Let $V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a continuous set-valued map and let $\mathcal{M} \subset \mathbb{R}^n$ be a closed set such that the following statements hold.

i) $V(\mathcal{F}(x)) \subseteq V(x)$ for every $x \in \mathbb{R}^n$.

ii) If $V(y) = V(x)$ for some $y \in \mathcal{F}(x)$, then $x \in \mathcal{M}$.

Then every bounded solution $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$ of (6.22) converges to $\mathcal{M}$, that is, $\lim_{k\to\infty} \operatorname{dist}(x(k), \mathcal{M}) = 0$.

Proof. It follows from *i*) that $V(\psi(k+1)) \subseteq V(\psi(k))$ for every solution $\psi(k)$, $k \in \overline{\mathbb{Z}}_+$, of (6.22). Thus, the sequence of closed sets $\{V(\psi(k)\}_{k=0}^{\infty}$ is nonincreasing, and hence, $\lim_{k\to\infty} V(\psi(k)) = \cap_{k=0}^{\infty} V(\psi(k)) \triangleq \mathcal{V}$ [62]. Next, note that since $\psi(k)$, $k \in \overline{\mathbb{Z}}_+$, is bounded, $\Omega(\psi)$ is nonempty. Now, for all $x \in \Omega(\psi)$, it follows from the definition of $\Omega(\psi)$ and the continuity of V that $V(x) = \mathcal{V}$. Moreover, the outer semicontinuity of $\mathcal{F}$ ensures that $\Omega(\psi)$ is weakly positively (and negatively) invariant. Specifically, for every $x \in \Omega(\psi)$, there exists $y \in \mathcal{F}(x)$ such that $y \in \Omega(\psi)$. Thus, for every $x \in \Omega(\psi)$, there exists $y \in \mathcal{F}(x)$ such that $V(x) = V(y) = \mathcal{V}$, and hence, $\Omega(\psi) \subseteq \mathcal{M}$. Finally, since $\operatorname{dist}(\psi(k), \omega(\psi)) \to 0$ as $k \to 0$, it follows that $\psi(k) \to \mathcal{M}$ as $k \to \infty$. □

Next, we illustrate Theorem 6.5.1 by applying it to the network system given by (6.19). The conclusions of the proposition below are weaker than the results obtained directly in Section 6.4. However, the approach can prove beneficial for nonlinear network architectures where direct computation relying on a linear structure is not possible as well as for partial graph connectivity structures with directed information flow.

Proposition 6.5.1. Consider a network of N agents with an all-to-all graph connectivity given by (6.19) and let $x(\cdot)$ be a bounded solution of (6.19). Then, $\limsup_{k\to\infty} \|x_i(k) - x_j(k)\|_2 \leq 4r$ for every $i, j = 1, \ldots, N$.

Proof. Let the set-valued map $V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be given by

$$V(x) = \mathcal{B}_{\delta_1(x)}(x_{\rm ave}) \times \cdots \times \mathcal{B}_{\delta_N(x)}(x_{\rm ave}),$$

where, for $i \in \{1, \ldots, N\}$,

$$\delta_i(x) = \begin{cases} \|x_i - x_{\rm ave}\|_2, & \|x_i - x_{\rm ave}\|_2 \geq 2r, \\ 2r, & \|x_i - x_{\rm ave}\|_2 \leq 2r, \end{cases}$$

and "$\times$" denotes Cartesian product. Note that V is continuous and has closed and bounded values. Next, it can be shown using a similar argument as in Section 6.4 that

$$x_i(k+1) - x_{\rm ave}(k+1) \in \mathcal{B}_{\alpha r}(\alpha x_{\rm ave}(k)) - \mathcal{B}_{\alpha r}(x_{\rm ave}(k)) + (1-\alpha)x_i(k), \quad k \in \overline{\mathbb{Z}}_+,$$

which implies

$$\|x_i(k+1) - x_{\rm ave}(k+1)\|_2 \leq (1-\alpha)\|x_i(k) - x_{\rm ave}(k)\|_2 + 2r\alpha.$$

Hence, the function $\delta_i(\cdot)$ decreases for $\|x_i - x_{\rm ave}\|_2 > 2r$ and remains constant for $\|x_i - x_{\rm ave}\|_2 \leq 2r$, $i \in \{1, \ldots, N\}$, and hence, Conditions i) and ii) of Theorem 6.5.1 are satisfied. Now, it follows from Theorem 6.5.1 that every bounded solution $x_i(\cdot)$, $i \in \{1, \ldots, N\}$, converges to $\mathcal{B}_{2r}(x_{\rm ave})$. Hence, $\|x_i(k) - x_j(k)\|_2 \leq 4r$ as $k \to \infty$ for all $i, j = 1, \ldots, N$. □

6.6. Illustrative Numerical Examples

In this section, we present several numerical examples to demonstrate the efficacy of the proposed framework. Specifically, we consider a random network of 10 agents

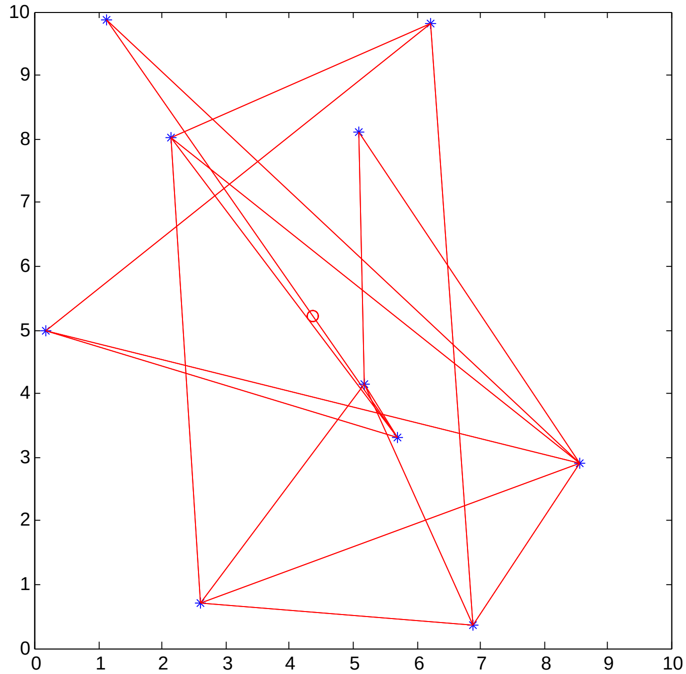

Figure 6.2. Initial network configuration of 10 agents with sensor accuracy of radius $r = 1$.

with different network topologies and agent dynamics given by (6.4). Furthermore, we assume that the ith agent uncertainty is modeled as a standard white noise process.

Figures 6.2, 6.3, 6.4, and 6.5 show the initial, intermediate, and final network configurations, as well as $\|x(t) - \mathbf{e}_N \overline{x}\|_2$ versus time, of the network of agents when agents have sensor accuracy of radius 1, $\lambda_2(\mathcal{L}) = 1.5568$, and $\lambda_N(\mathcal{L}) = 7.5704$. The circle indicates the location of the initial centroid of the agents. Note that

$$\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}\|_2 \leq \frac{\lambda_N(\mathcal{L})\sqrt{N}r}{\lambda_2(\mathcal{L})} = 15.3775.$$

Alternatively, Figures 6.6, 6.7, 6.8, and 6.9 show the initial, intermediate, and final network configurations, as well as $\|x(t) - \mathbf{e}_N \overline{x}\|_2$ versus time, of the network of agents when agents have sensor accuracy of radius 1, $\lambda_2(\mathcal{L}) = 0.1172$, and $\lambda_N(\mathcal{L}) = 4.3721$. Once again, the circle indicates the location of the initial centroid of the agents. Note

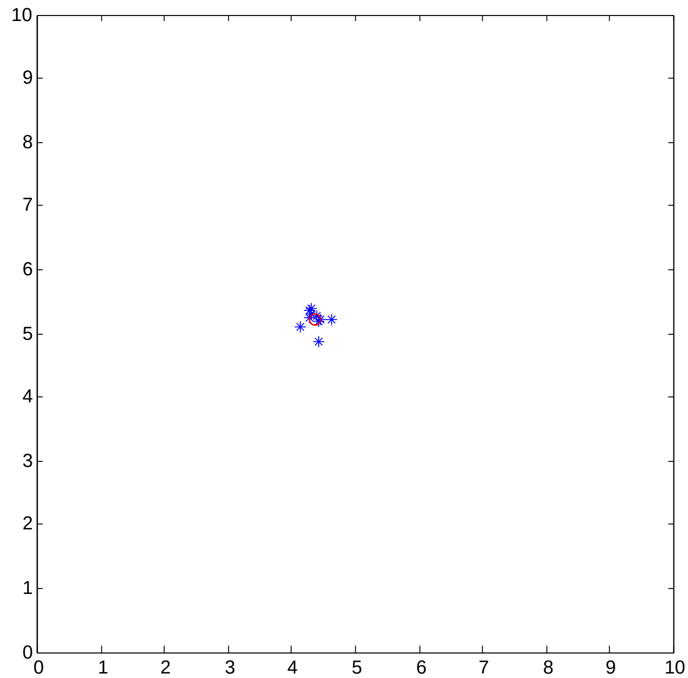

Figure 6.3. Network configuration of 10 agents with sensor accuracy of radius $r = 1$ at $t = 3.5$ sec.

that

$$\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}\|_2 \leq \frac{\lambda_N(\mathcal{L})\sqrt{N}r}{\lambda_2(\mathcal{L})} = 117.9675.$$

Finally, Figures 6.10, 6.11, and 6.12 show the initial, intermediate, and final configurations, respectively, of the network of 10 agents when agents have sensor accuracy of radius 0.5 and the network is all-to-all connected. The simulation shows that the agents reach a consensus set with diameter less than $2r = 1$. The circle indicates a set with diameter 1 centered at the initial centroid of the agents.

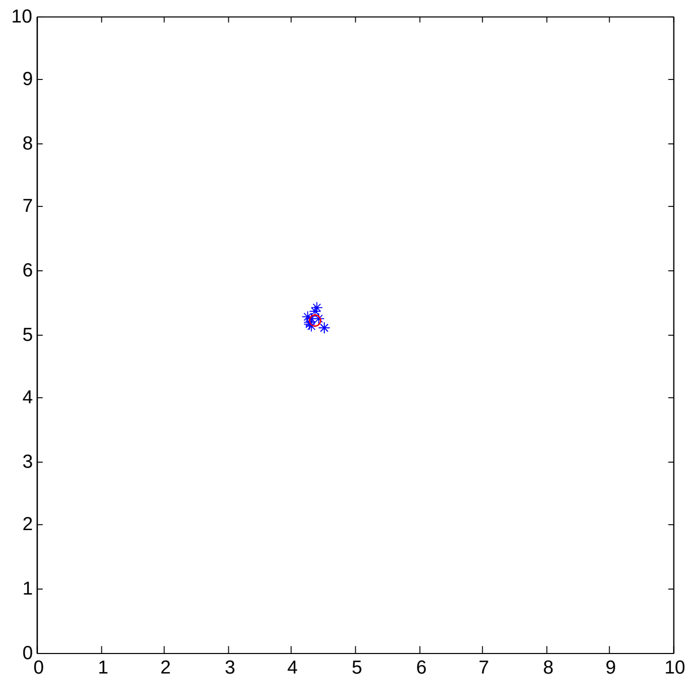

Figure 6.4. Network configuration of 10 agents with sensor accuracy of radius $r = 1$ at $t = 7.5$ sec.

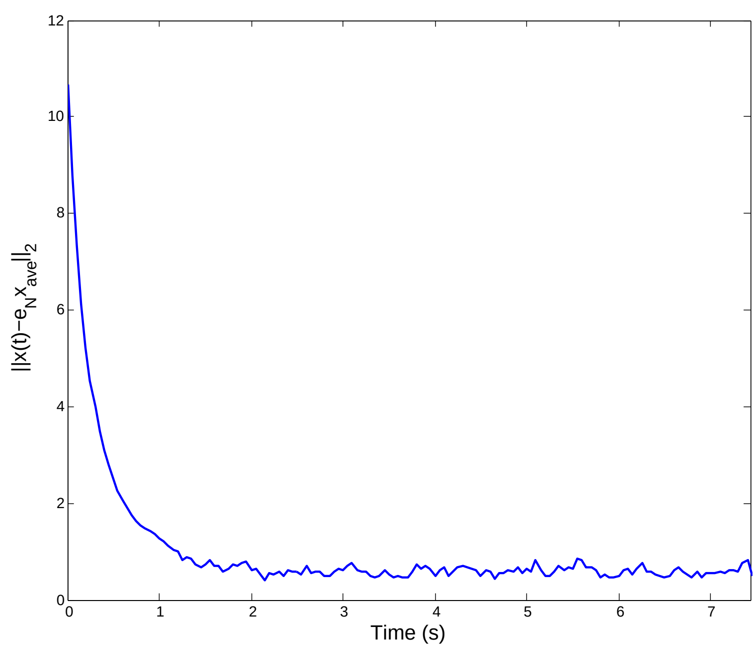

Figure 6.5. Plot of $\|x(t) - \mathbf{e}_N \overline{x}\|_2$ versus time.

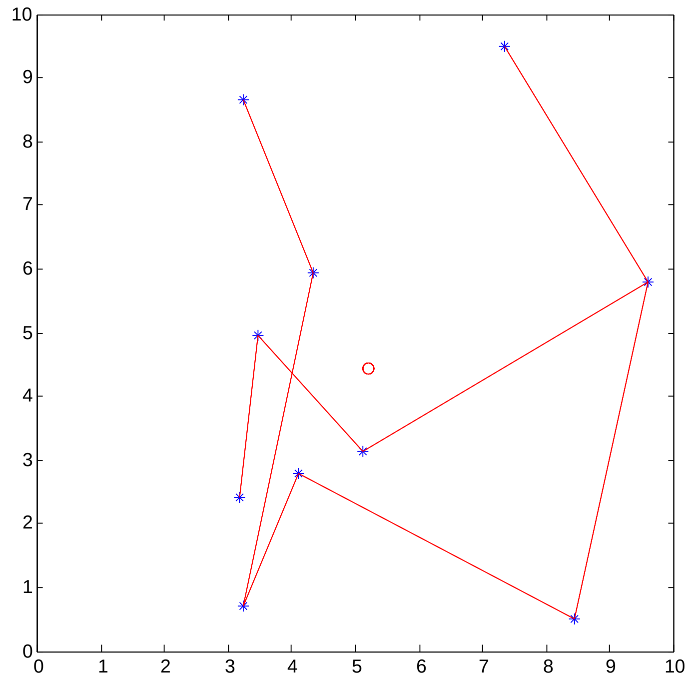

Figure 6.6. Initial network configuration of 10 agents with sensor accuracy of radius $r = 1$.

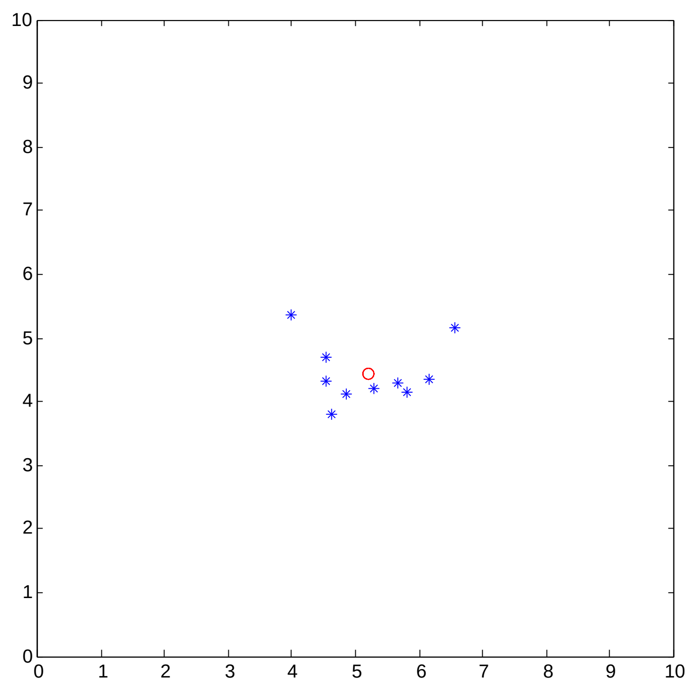

Figure 6.7. Network configuration of 10 agents with sensor accuracy of radius $r = 1$ at $t = 3.5$ sec.

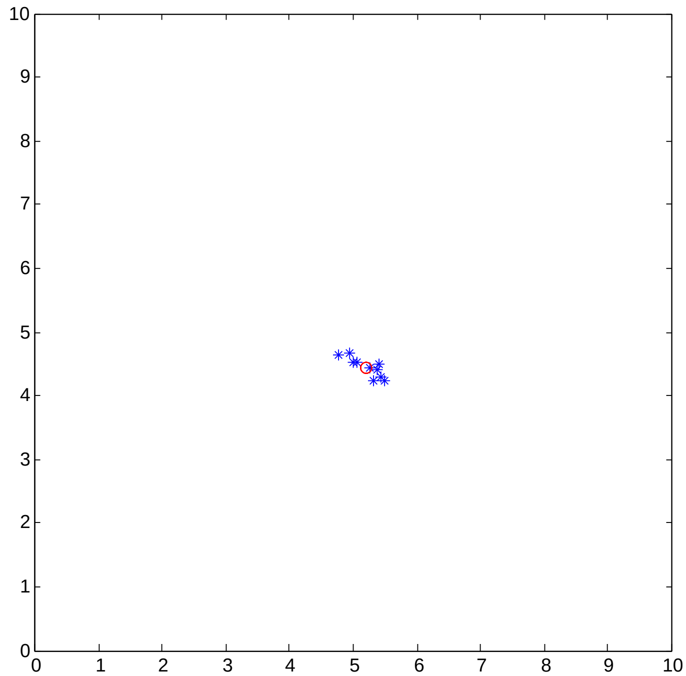

Figure 6.8. Network configuration of 10 agents with sensor accuracy of radius $r = 1$ at $t = 7.5$ sec.

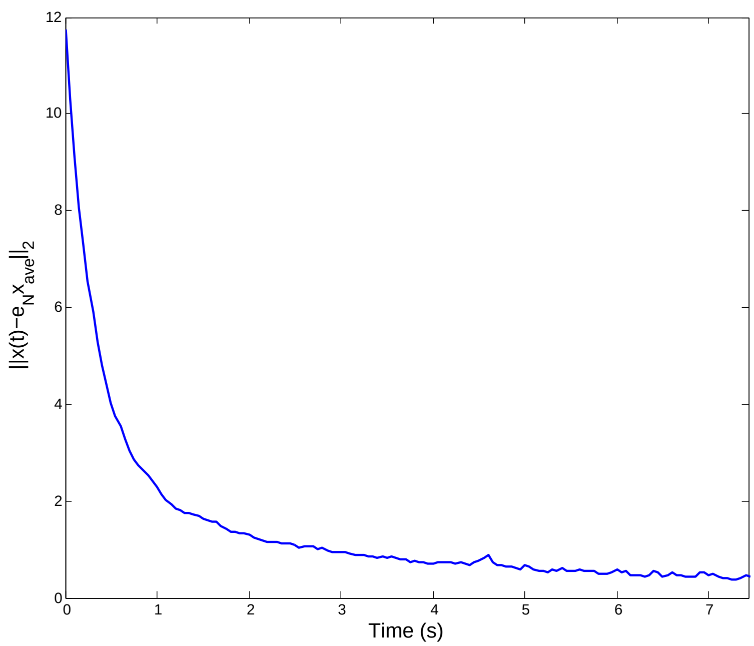

Figure 6.9. Plot of $\|x(t) - \mathbf{e}_N \overline{x}\|_2$ versus time.

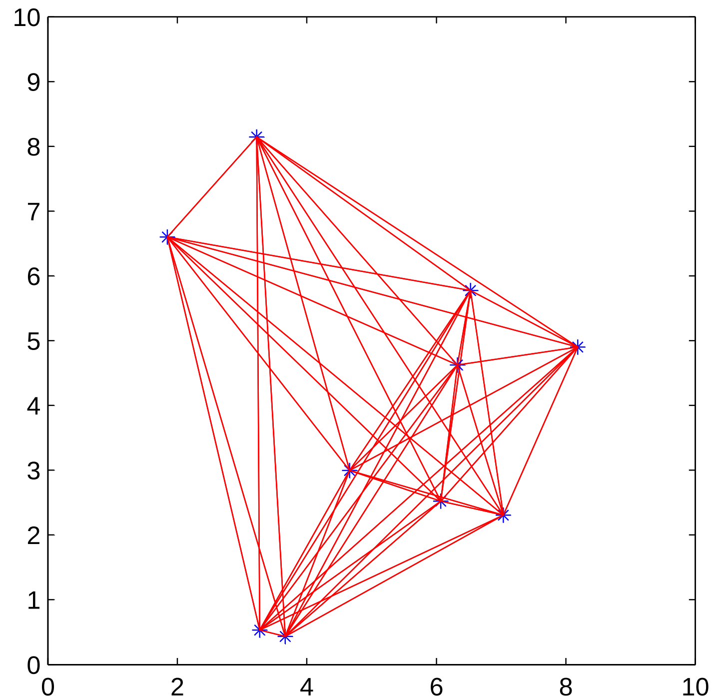

Figure 6.10. Initial network configuration of 10 agents with sensor accuracy of radius $r = 0.5$.

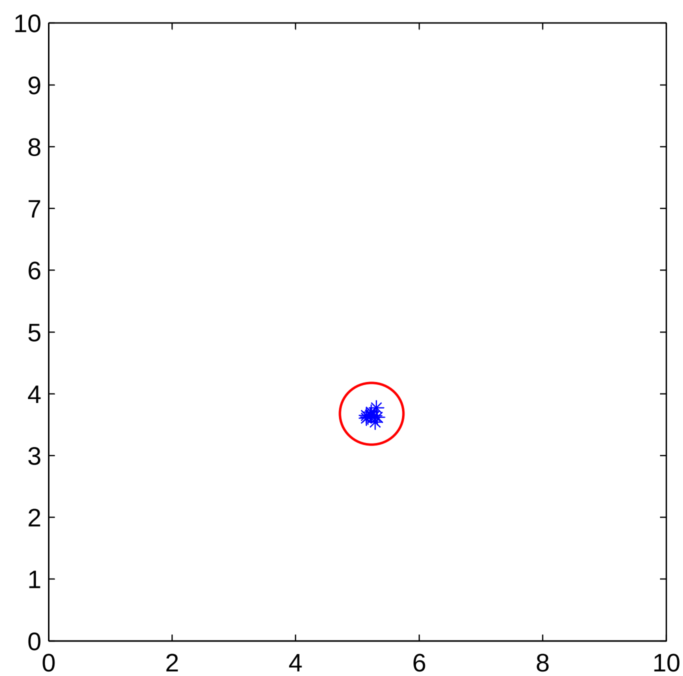

Figure 6.11. Network configuration of 10 agents with sensor accuracy of radius $r = 0.5$ at $t = 3.5$ sec.

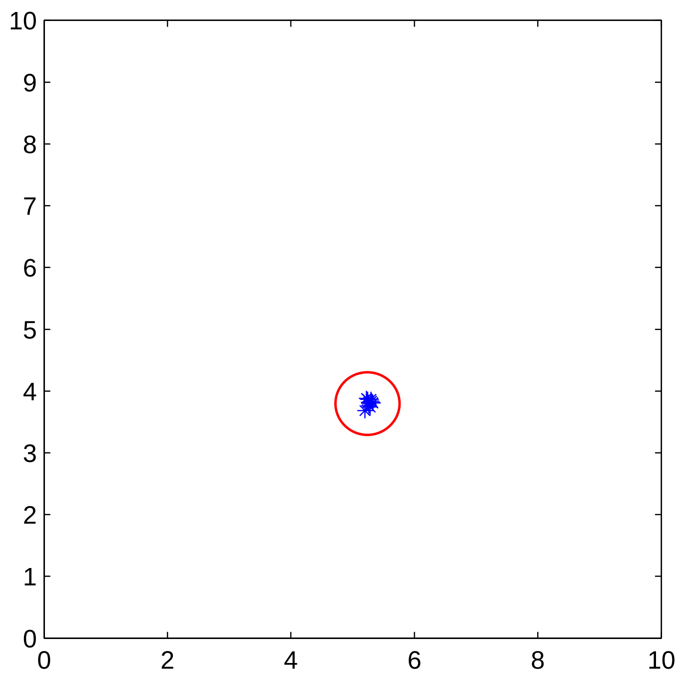

Figure 6.12. Network configuration of 10 agents with sensor accuracy of radius $r = 0.5$ at $t = 7.5$ sec.

Chapter 7

Adaptive Estimation using Multiagent Network Identifiers with Undirected and Directed Graph Topologies

7.1. Introduction

In this chapter, we consider the problem of adaptive estimation of a linear system with unknown plant and input matrices. In particular, we propose a novel distributed observer architecture that adaptively identifies the dynamic system matrices using a group of N agents. Each agent generates its own adaptive identifier which is based on the identifier architecture presented in [53]. Furthermore, it is shown that if the adaptive identifiers have the same structure, but do not share information (i.e., are not connected), then there is no guarantee that the N adaptive identifiers will have their estimates converge to the same value without a persistency of excitation condition being imposed. Alternatively, when the update laws for the parameter identifiers are modified to include interagent information exchange, then consensus of both the state and parameter estimates are guaranteed, and thus, emulating a persistency of excitation condition.

The proposed adaptive identifier architecture includes additional terms in both the state and parameter equations, which effectively penalize the mismatch between all estimates and take the form of nonnegative damping terms that serve to enhance the convergence properties of the state and parameter errors. The adaptive estimation architecture builds on the work of [71] on adaptive consensus control of multiagent systems with the key difference being that the mismatch between the state and parameter estimates is also penalized, and thus, accounting for interagent communication constraints.

For nonadaptive estimators, a linear estimator scheme that considers a penalized mismatch of the parameter estimates was proposed in [19, 70]. Alternatively, within the context of distributed Kalman filtering for sensor networks, agreement of the state and parameter estimates, as a measure that is independent of the network topology and wherein the deviations of the parameter estimates are measured from their mean, was considered in [54]. Distributed adaptive control for convergence using consensus learning of sensory information for networked robots is addressed in [65, 66].

The added benefit of the proposed network architecture of the adaptive identifiers, which penalize the mismatch between *both* state and parameter estimates, is the abstract form that the collective error dynamics take. In particular, the proposed framework allows one to decouple the graph connectivity (i.e., the graph Laplacian) from the stability analysis of the parameter errors by simply replacing a nonnegative damping-like matrix representing the connectivity of the graph topology with another matrix representing a more general interagent connectivity. Finally, we note that the proposed adaptive estimation multiagent network identifier framework was first explored in [21] for the very restrictive case of all-to-all graph connectivity. In this chapter, we extend the results of [21] to develop adaptive multiagent network identifiers with undirected and directed graph topologies.

7.2. Adaptive Estimation Problem

In this section, we present a brief exposition of the standard centralized adaptive estimation for plant parameter estimation in dynamical systems involving full state information. Specifically, we consider dynamical systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{7.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector and $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input. Here, we assume that the plant and input matrices A and B are unknown, and the state $x(t)$ and control input signal $u(t)$ are bounded for all $t \geq 0$. To identify the matrices A and B online, we consider the adaptive observer given by ([43])

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{A}(t)x(t) + \hat{B}(t)u(t) + A_{\rm m}(\hat{x}(t) - x(t)) \\ &= A_{\rm m}\hat{x}(t) + (\hat{A}(t) - A_{\rm m})x(t) + \hat{B}(t)u(t), \quad \hat{x}(0) = \hat{x}_0 \neq x_0, \quad t \geq 0, \end{aligned} \tag{7.2}$$

where $\hat{x}(t) \in \mathbb{R}^n$, $t \geq 0$, is the observer state, $\hat{A}(t) \in \mathbb{R}^{n\times n}$, $t \geq 0$, is the adaptive estimate of A, and $\hat{B}(t) \in \mathbb{R}^{n\times m}$, $t \geq 0$, is the adaptive estimate of B. The matrix $A_{\rm m} \in \mathbb{R}^{n\times n}$ is a design matrix that is Hurwitz and defines the observer poles.

To establish online estimates for the system matrices A and B, define the state and parameter errors, respectively, by $e(t) \triangleq \hat{x}(t) - x(t)$, $t \geq 0$, $\tilde{A}(t) \triangleq \hat{A}(t) - A$, $t \geq 0$, and $\tilde{B}(t) \triangleq \hat{B}(t) - B$, $t \geq 0$. Then, the system error dynamics and parameter update dynamics are given by

$$\dot{e}(t) = A_{\rm m}e(t) + \tilde{A}(t)x(t) + \tilde{B}(t)u(t), \quad e(0) = e_0, \quad t \geq 0, \tag{7.3}$$

$$\dot{\hat{A}}(t) = -\Gamma_{\rm a}Pe(t)x^{\rm T}(t), \quad \hat{A}(0) = \hat{A}_0 \neq A, \tag{7.4}$$

$$\dot{\hat{B}}(t) = -\Gamma_{\rm b}Pe(t)u^{\rm T}(t), \quad \hat{B}(0) = \hat{B}_0 \neq B, \tag{7.5}$$

where $\Gamma_{\rm a} \in \mathbb{R}^{n\times n}$ and $\Gamma_{\rm b} \in \mathbb{R}^{m\times m}$ are positive-definite gain matrices and $P \in \mathbb{R}^{n\times n}$ is a positive-definite solution of the Lyapunov equation

$$0 = A_{\mathrm{m}}^{\mathrm{T}} P + P A_{\mathrm{m}} + R, \tag{7.6}$$

where $R \in \mathbb{R}^{n \times n}$ is a given positive-definite matrix. Since A_m is Hurwitz, it follows from converse Lyapunov theory [31] that there exists a unique positive-definite matrix $P \in \mathbb{R}^{n \times n}$ satisfying (7.6) for a given positive definite matrix $R \in \mathbb{R}^{n \times n}$. The adaptive update laws for $\hat{A}(t)$, $t \geq 0$, and $\hat{B}(t)$, $t \geq 0$, given by (7.4) and (7.5), respectively, can be derived using standard Lyapunov analysis by considering the Lyapunov function candidate

$$V(e, \tilde{A}, \tilde{B}) = e^{\mathrm{T}} P e + \operatorname{tr} \tilde{A}^{\mathrm{T}} \Gamma_{\mathrm{a}}^{-1} \tilde{A} + \operatorname{tr} \tilde{B}^{\mathrm{T}} \Gamma_{\mathrm{b}}^{-1} \tilde{B}. \tag{7.7}$$

Note that $V(0,0,0) = 0$ and $V(e, \tilde{A}, \tilde{B}) > 0$ for all $(e, \tilde{A}, \tilde{B}) \neq (0,0,0)$. Now, differentiating (7.7) along the trajectories of (7.3)–(7.5) yields

$$\begin{aligned}
&\dot{V}(e(t), \tilde{A}(t), \tilde{B}(t)) \\
&\quad = \dot{e}^{\mathrm{T}}(t) P e(t) + e^{\mathrm{T}}(t) P \dot{e}(t) + 2 \operatorname{tr} \tilde{A}^{\mathrm{T}}(t) \Gamma_{\mathrm{a}}^{-1} \dot{\hat{A}}(t) + 2 \operatorname{tr} \tilde{B}^{\mathrm{T}}(t) \Gamma_{\mathrm{b}}^{-1} \dot{\hat{B}}(t) \\
&\quad = e^{\mathrm{T}}(t)(A_m^{\mathrm{T}} P + P A_m) e(t) + 2 \operatorname{tr} \tilde{A}^{\mathrm{T}}(t) P e(t) x^{\mathrm{T}}(t) \\
&\qquad + 2 \operatorname{tr} \tilde{B}^{\mathrm{T}}(t) P e(t) u^{\mathrm{T}}(t) + 2 \operatorname{tr} \tilde{A}^{\mathrm{T}}(t) \Gamma_{\mathrm{a}}^{-1} \dot{\hat{A}}(t) + 2 \operatorname{tr} \tilde{B}^{\mathrm{T}}(t) \Gamma_{\mathrm{b}}^{-1} \dot{\hat{B}}(t).
\end{aligned} \tag{7.8}$$

Using the update laws (7.4) and (7.5) in (7.8), it follows that

$$\dot{V}(e(t), \tilde{A}(t), \tilde{B}(t)) = -e^{\mathrm{T}}(t) R e(t) \leq 0, \quad t \geq 0, \tag{7.9}$$

which guarantees that the error signal $e(t)$, $t \geq 0$, and parameter errors $\tilde{A}(t)$, $t \geq 0$, and $\tilde{B}(t)$, $t \geq 0$, are Lyapunov stable, and hence, are bounded for all $t \geq 0$. Since $e(t)$, $t \geq 0$, $\hat{A}(t)$, $t \geq 0$, $\hat{B}(t)$, $t \geq 0$, $x(t)$, $t \geq 0$, and $u(t)$, $t \geq 0$, are bounded for all $t \geq 0$, it follows that $\dot{e}(t)$, $t \geq 0$, is bounded, and hence, $\ddot{V}(e(t), \tilde{A}(t), \tilde{B}(t))$ is bounded for all $t \geq 0$. Now, it follows from Barbalat's lemma [31, p. 221] that $\dot{V}(e(t), \tilde{A}(t), \tilde{B}(t)) \to 0$ as $t \to \infty$, and hence, $e(t)$ converges to zero asymptotically. Convergence of the adaptive estimates to their true values can be shown when a persistency of excitation condition is imposed [43, 53].

The system error dynamics (7.3) and parameter dynamics (7.4) and (7.5) can be written in operator form as

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\tilde{A}}(t) \\ \dot{\tilde{B}}(t) \end{bmatrix} = \mathfrak{A}(x(t), u(t)) \begin{bmatrix} e(t) \\ \tilde{A}(t) \\ \tilde{B}(t) \end{bmatrix}, \quad \begin{bmatrix} e(0) \\ \tilde{A}(0) \\ \tilde{B}(0) \end{bmatrix} = \begin{bmatrix} e_0 \\ \tilde{A}_0 \\ \tilde{B}_0 \end{bmatrix}, \quad t \geq 0, \tag{7.10}$$

where

$$\mathfrak{A}(x(t), u(t)) \triangleq \left[\begin{array}{c|cc} A_{\mathrm{m}} & (\cdot)x(t) & (\cdot)u(t) \\ \hline -\Gamma_{\mathrm{a}} P(\cdot) x^{\mathrm{T}}(t) & 0 & 0 \\ -\Gamma_{\mathrm{b}} P(\cdot) u^{\mathrm{T}}(t) & 0 & 0 \end{array}\right].$$

The structure given in (7.10) involving the skew-adjoint, state-dependent operator $\mathfrak{A}(\cdot, \cdot)$ is characteristic of adaptive systems [50]. The same structure is observed in the case of distributed adaptive consensus identifiers presented in Section 7.3, in which the operator form involves the same structure as above with additional terms arising due to consensus enforcement. As we see in Section 7.3, this form can be related to the Laplacian of the graph topology of the network.

7.3. Adaptive Distributed Observers

In this section, we consider a distributed adaptive observer problem for (7.1). Specifically, we consider N *noninteracting* agents given by

$$\dot{\hat{x}}_i(t) = A_{\mathrm{m}} \hat{x}_i(t) + (\hat{A}_i(t) - A_{\mathrm{m}}) x(t) + \hat{B}_i(t) u(t), \quad \hat{x}_i(0) = \hat{x}_{i0} \neq x(0), \quad t \geq 0, \tag{7.11}$$

$$\dot{\hat{A}}_i(t) = -\Gamma_{\mathrm{a}i} P e_i(t) x^{\mathrm{T}}(t), \quad \hat{A}_i(0) = \hat{A}_{i0}, \tag{7.12}$$

$$\dot{\hat{B}}_i(t) = -\Gamma_{\mathrm{b}i} P e_i(t) u^{\mathrm{T}}(t), \quad \hat{B}_i(0) = \hat{B}_{i0}, \tag{7.13}$$

where, for $i = 1, \ldots, N$, $\hat{x}_i(t) \in \mathbb{R}^n$, $t \geq 0$, $\hat{A}_i(t) \in \mathbb{R}^{n \times n}$, $t \geq 0$, $\hat{B}_i(t) \in \mathbb{R}^{n \times m}$, $t \geq 0$, $e_i(t) \triangleq \hat{x}_i(t) - x(t)$, $t \geq 0$, and $\Gamma_{\mathrm{a}i} \in \mathbb{R}^{n \times n}$ and $\Gamma_{\mathrm{b}i} \in \mathbb{R}^{m \times m}$ are positive-definite gain matrices. Here, we can easily replace A_{m} in (7.11) with $A_{\mathrm{m}i}$, where $A_{\mathrm{m}i}$, $i = 1, \ldots, N$, are Hurwitz design matrices. In this case, the results in the remainder of the chapter hold with minor extensions.

To quantify a measure of disagreement between the state estimates and parameter estimates that is independent of the network topology, we use the deviation of these estimates from the mean defined by

$$\delta_{ie}(t) \triangleq \hat{x}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{x}_j(t) = e_i(t) - \frac{1}{N}\sum_{j=1}^{N}e_j(t), \tag{7.14}$$

$$\delta_{i\mathrm{a}}(t) \triangleq \hat{A}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{A}_j(t) = \tilde{A}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\tilde{A}_j(t), \tag{7.15}$$

$$\delta_{i\mathrm{b}}(t) \triangleq \hat{B}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{B}_j(t) = \tilde{B}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\tilde{B}_j(t), \tag{7.16}$$

for $i = 1, \ldots, N$. In this case, the pairwise disagreement is defined as

$$\hat{x}_{ij}(t) \triangleq \hat{x}_i(t) - \hat{x}_j(t) = e_{ij}(t) = e_i(t) - e_j(t), \tag{7.17}$$

$$\hat{A}_{ij}(t) \triangleq \hat{A}_i(t) - \hat{A}_j(t) = \tilde{A}_{ij}(t) = \tilde{A}_i(t) - \tilde{A}_j(t), \tag{7.18}$$

$$\hat{B}_{ij}(t) \triangleq \hat{B}_i(t) - \hat{B}_j(t) = \tilde{B}_{ij}(t) = \tilde{B}_i(t) - \tilde{B}_j(t), \tag{7.19}$$

for $i, j = 1, \ldots, N$, $i \neq j$.

Note that the distributed adaptive observers (7.11)–(7.13) can be placed in the form of (7.10). To see this, define

$$E(t) \triangleq \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_N(t) \end{bmatrix} \in \mathbb{R}^{nN},\ \hat{\mathbb{A}}(t) \triangleq \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ \vdots \\ \hat{A}_N(t) \end{bmatrix} \in \mathbb{R}^{nN\times n},\ \hat{\mathbb{B}}(t) \triangleq \begin{bmatrix} \hat{B}_1(t) \\ \hat{B}_2(t) \\ \vdots \\ \hat{B}_N(t) \end{bmatrix} \in \mathbb{R}^{nN\times m},$$

with $\tilde{\mathbb{A}}(t)$ and $\tilde{\mathbb{B}}(t)$ defined analogously, and define $\mathbb{A}_{\mathrm{m}} \triangleq I_N \otimes A_{\mathrm{m}}$, $\mathbb{P} \triangleq I_N \otimes P$,

$$\Gamma_{\mathrm{a}} \triangleq \begin{bmatrix} \Gamma_{\mathrm{a}1} & 0_{n\times n} & \ldots \\ \vdots & \ddots & \vdots \\ 0_{n\times n} & \ldots & \Gamma_{\mathrm{a}N} \end{bmatrix},\quad \Gamma_{\mathrm{b}} \triangleq \begin{bmatrix} \Gamma_{\mathrm{b}1} & 0_{n\times n} & \ldots \\ \vdots & \ddots & \vdots \\ 0_{n\times n} & \ldots & \Gamma_{\mathrm{b}N} \end{bmatrix}.$$

Then,

$$\dot{E}(t) = \mathbb{A}_{\rm m}E(t) + \tilde{\mathbb{A}}(t)x(t) + \tilde{\mathbb{B}}(t)u(t), \quad E(0) = E_0, \quad t \geq 0, \tag{7.20}$$

$$\dot{\tilde{\mathbb{A}}}(t) = -\Gamma_{\rm a}\mathbb{P}E(t)x^{\rm T}(t), \quad \tilde{\mathbb{A}}(0) = \tilde{\mathbb{A}}_0, \tag{7.21}$$

$$\dot{\tilde{\mathbb{B}}}(t) = -\Gamma_{\rm b}\mathbb{P}E(t)u^{\rm T}(t), \quad \tilde{\mathbb{B}}(0) = \tilde{\mathbb{B}}_0, \tag{7.22}$$

or, equivalently,

$$\begin{bmatrix} \dot{E}(t) \\ \dot{\tilde{\mathbb{A}}}(t) \\ \dot{\tilde{\mathbb{B}}}(t) \end{bmatrix} = \tilde{\mathfrak{A}}(x(t), u(t)) \begin{bmatrix} E(t) \\ \tilde{\mathbb{A}}(t) \\ \tilde{\mathbb{B}}(t) \end{bmatrix}, \quad \begin{bmatrix} E(0) \\ \tilde{\mathbb{A}}(0) \\ \tilde{\mathbb{B}}(0) \end{bmatrix} = \begin{bmatrix} E_0 \\ \tilde{\mathbb{A}}_0 \\ \tilde{\mathbb{B}}_0 \end{bmatrix}, \quad t \geq 0, \tag{7.23}$$

where

$$\tilde{\mathfrak{A}}(x(t), u(t)) \triangleq \left[\begin{array}{c|cc} \mathbb{A}_{\rm m} & (\cdot)x(t) & (\cdot)u(t) \\ \hline -\Gamma_{\rm a}\mathbb{P}(\cdot)x^{\rm T}(t) & 0 & 0 \\ -\Gamma_{\rm b}\mathbb{P}(\cdot)u^{\rm T}(t) & 0 & 0 \end{array}\right].$$

Equation (7.23) is the multiagent identifier version of (7.10) and shows that the distributed adaptive observers (7.11)–(7.13) have identical stability and convergence properties as (7.2), (7.4), and (7.5).

In particular, consider the distributed Lyapunov function candidates for each agent given by

$$V_i(e_i, \tilde{A}_i, \tilde{B}_i) = e_i^{\rm T}Pe_i + {\rm tr}\,\tilde{A}_i^{\rm T}\Gamma_{{\rm a}i}^{-1}\tilde{A}_i + {\rm tr}\,\tilde{B}_i^{\rm T}\Gamma_{{\rm b}i}^{-1}\tilde{B}_i, \quad i = 1, \ldots, N. \tag{7.24}$$

Now, the stability of the collective dynamics of (7.11)–(7.13) can be established using the Lyapunov function candidate

$$V(E, \tilde{\mathbb{A}}, \tilde{\mathbb{B}}) = \sum_{i=1}^{N} V_i(e_i, \tilde{A}_i, \tilde{B}_i). \tag{7.25}$$

Specifically, differentiating (7.25) along the trajectories of (7.11)–(7.13) yields

$$\dot{V}_i(e_i(t), \tilde{A}_i(t), \tilde{B}_i(t)) = -e_i^{\rm T}(t)Re_i(t), \quad t \geq 0, \quad i = 1, \ldots, N,$$

and hence,

$$\dot{V}(E(t), \tilde{\mathbb{A}}(t), \tilde{\mathbb{B}}(t)) = -\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) R e_i(t) = -E^{\mathrm{T}}(t)\mathbb{R}E(t) \leq 0, \quad t \geq 0,$$

where $\mathbb{R} \triangleq I_N \otimes R$. Now, similar arguments as in Section 7.2 can be used to show that $E(t)$ converges to zero asymptotically.

7.4. Adaptive Consensus of Distributed Observers over Networks with Undirected Graph Topologies

In this section, we consider a multiagent system in which N agents are utilized to adaptively estimate the plant parameters A and B over a connected undirected network. Each agent provides its own estimate $\hat{A}_i(t)$, $t \geq 0$, and $\hat{B}_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, and strives to arrive at common estimates, that is, reach consensus on the parameter adaptive estimates. The update laws for the parameter identifiers given by (7.11) are modified to include interagent communication with a penalty on the mismatch between the parameter estimates $\hat{A}_i(t)$, $t \geq 0$, and $\hat{B}_i(t)$, $t \geq 0$. Even though the individual adaptive estimates require a condition of persistency of excitation to ensure parameter convergence, the proposed adaptive consensus modification guarantees that all the parameter estimates agree with each other, which emulates a persistency of excitation condition.

Theorem 7.4.1. Consider the dynamical system (7.1) with A and B unknown. Assume that $\mathfrak{G}$ defines a connected undirected graph of N agents implementing the distributed adaptive observers given by

$$\dot{\hat{x}}_i(t) = A_{\mathrm{m}}\hat{x}_i(t) + (\hat{A}_i(t) - A_{\mathrm{m}})x(t) + \hat{B}_i(t)u(t) - P^{-1}\sum_{j\in\mathcal{N}(i)}(\hat{x}_i(t) - \hat{x}_j(t)),$$
$$\hat{x}_i(0) = \hat{x}_{i0}, \quad t \geq 0, \quad (7.26)$$

$$\dot{\hat{A}}_i(t) = -\Gamma_{\mathrm{a}i}Pe_i(t)x^{\mathrm{T}}(t) - \Gamma_{\mathrm{a}i}\sum_{j\in\mathcal{N}(i)}(\hat{A}_i(t) - \hat{A}_j(t)), \quad \hat{A}_i(0) = \hat{A}_{i0}, \quad (7.27)$$

$$\dot{\hat{B}}_i(t) = -\Gamma_{\mathrm{b}i} P e_i(t) u^{\mathrm{T}}(t) - \Gamma_{\mathrm{b}i} \sum_{j\in\mathcal{N}(i)} (\hat{B}_i(t) - \hat{B}_j(t)), \quad \hat{B}_i(0) = \hat{B}_{i0}, \tag{7.28}$$

where $i = 1,\ldots,N$ and P satisfies (7.6). Then, the solution $(E(t), \tilde{\mathbb{A}}(t), \tilde{\mathbb{B}}(t))$ of the parameter error system is Lyapunov stable for all $(E_0, \tilde{\mathbb{A}}_0, \tilde{\mathbb{B}}_0) \in \mathbb{R}^{nN} \times \mathbb{R}^{nN\times n} \times \mathbb{R}^{nN\times m}$ and $t \geq 0$, and $\lim_{t\to\infty} \tilde{x}_{ij}(t) = 0$, $\lim_{t\to\infty} \hat{A}_{ij}(t) = 0$, $\lim_{t\to\infty} \hat{B}_{ij}(t) = 0$, $i,j = 1,\ldots,N$, and $\lim_{t\to\infty} e_i(t) = 0$, $i = 1,\ldots,N$.

Proof. Given (7.26)–(7.28) the state and parameter error dynamics are given by

$$\dot{e}_i(t) = A_{\mathrm{m}} e_i(t) + \tilde{A}_i(t)x(t) + \tilde{B}_i(t)u(t) - P^{-1} \sum_{j\in\mathcal{N}(i)} e_{ij}(t), \quad e_i(0) = e_{i0}, \quad t \geq 0, \tag{7.29}$$

$$\dot{\tilde{A}}_i(t) = -\Gamma_{\mathrm{a}i} P e_i(t) x^{\mathrm{T}}(t) - \Gamma_{\mathrm{a}i} \sum_{j\in\mathcal{N}(i)} \tilde{A}_{ij}(t), \quad \tilde{A}_i(0) = \tilde{A}_{i0}, \tag{7.30}$$

$$\dot{\tilde{B}}_i(t) = -\Gamma_{\mathrm{b}i} P e_i(t) u^{\mathrm{T}}(t) - \Gamma_{\mathrm{b}i} \sum_{j\in\mathcal{N}(i)} \tilde{B}_{ij}(t), \quad \tilde{B}_i(0) = \tilde{B}_{i0}, \tag{7.31}$$

for $i = 1,\ldots,N$. Next, consider the distributed Lyapunov function candidates given by (7.24) and note that the derivatives of $V_i(e_i, \tilde{A}_i, \tilde{B}_i)$, $i = 1,\ldots,N$, along the trajectories of (7.29)–(7.31) are given by

$$\begin{aligned}
&\dot{V}_i(e_i(t), \tilde{A}_i(t), \tilde{B}_i(t)) \\
&= e_i^{\mathrm{T}}(t)\left[A_{\mathrm{m}}^{\mathrm{T}} P + P A_{\mathrm{m}}\right] e_i(t) - 2e_i^{\mathrm{T}}(t) \sum_{j\in\mathcal{N}(i)} e_{ij}(t) + 2e_i^{\mathrm{T}}(t) P \tilde{A}_i(t) x(t) \\
&\quad + 2e_i^{\mathrm{T}}(t) P \tilde{B}_i(t) u(t) + 2\,\mathrm{tr}\Big[\dot{\tilde{A}}_i^{\mathrm{T}}(t) \Gamma_{\mathrm{a}i}^{-1} \tilde{A}_i(t)\Big] + 2\,\mathrm{tr}\Big[\dot{\tilde{B}}_i^{\mathrm{T}}(t) \Gamma_{\mathrm{b}i}^{-1} \tilde{B}_i(t)\Big] \\
&= -e_i^{\mathrm{T}}(t) R e_i(t) - 2e_i^{\mathrm{T}}(t) \sum_{j\in\mathcal{N}(i)} e_{ij}(t) + 2\,\mathrm{tr}\Big[\tilde{A}_i^{\mathrm{T}}(t)[P e_i(t) x^{\mathrm{T}}(t) + \Gamma_{\mathrm{a}i}^{-1} \dot{\tilde{A}}_i(t)]\Big] \\
&\quad + 2\,\mathrm{tr}\Big[\tilde{B}_i^{\mathrm{T}}(t)[P e_i(t) u^{\mathrm{T}}(t) + \Gamma_{\mathrm{b}i}^{-1} \dot{\tilde{B}}_i(t)]\Big] \\
&= -e_i^{\mathrm{T}}(t) R e_i(t) - 2e_i^{\mathrm{T}}(t) \sum_{j\in\mathcal{N}(i)} e_{ij}(t) - 2\,\mathrm{tr}\left[\tilde{A}_i^{\mathrm{T}}(t) \sum_{j\in\mathcal{N}(i)} \tilde{A}_{ij}(t)\right] \\
&\quad - 2\,\mathrm{tr}\left[\tilde{B}_i^{\mathrm{T}}(t) \sum_{j\in\mathcal{N}(i)} \tilde{B}_{ij}(t)\right].
\end{aligned} \tag{7.32}$$

Now, using (7.32), it follows from (7.25) that the derivative of $V(E,\tilde{\mathbb{A}},\tilde{\mathbb{B}})$ along error trajectories of (7.29)–(7.31) is given by

$$\begin{aligned}
&\dot{V}(E(t),\tilde{\mathbb{A}}(t),\tilde{\mathbb{B}}(t))\\
&=\sum_{i=1}^{N}\dot{V}_i(e_i(t),\tilde{A}_i(t),\tilde{B}_i(t))\\
&=-\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)Re_i(t)-2\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\sum_{j\in\mathcal{N}(i)}(e_i(t)-e_j(t))\\
&\quad-2\,\mathrm{tr}\left[\sum_{i=1}^{N}\tilde{A}_i^{\mathrm{T}}(t)\sum_{j\in\mathcal{N}(i)}(\tilde{A}_i(t)-\tilde{A}_j(t))\right]-2\,\mathrm{tr}\left[\sum_{i=1}^{N}\tilde{B}_i^{\mathrm{T}}(t)\sum_{j\in\mathcal{N}(i)}(\tilde{B}_i(t)-\tilde{B}_j(t))\right]\\
&=-\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)Re_i(t)-\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}\|e_{ij}(t)\|_2^2-\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}\|\tilde{A}_{ij}(t)\|_{\mathrm{F}}^2\\
&\quad-\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}\|\tilde{B}_{ij}(t)\|_{\mathrm{F}}^2\\
&\le 0,\quad t\ge 0,
\end{aligned}\tag{7.33}$$

where in (7.33) we used the identities

$$2\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\sum_{j\in\mathcal{N}(i)}(e_i(t)-e_j(t))=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}\|e_i(t)-e_j(t)\|_2^2,\tag{7.34}$$

$$2\,\mathrm{tr}\sum_{i=1}^{N}\tilde{A}_i^{\mathrm{T}}(t)\sum_{j\in\mathcal{N}(i)}(\tilde{A}_i(t)-\tilde{A}_j(t))=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}\mathrm{tr}[\tilde{A}_i(t)-\tilde{A}_j(t)][\tilde{A}_i(t)-\tilde{A}_j(t)]^{\mathrm{T}},\tag{7.35}$$

$$2\,\mathrm{tr}\sum_{i=1}^{N}\tilde{B}_i^{\mathrm{T}}(t)\sum_{j\in\mathcal{N}(i)}(\tilde{B}_i(t)-\tilde{B}_j(t))=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}\mathrm{tr}[\tilde{B}_i(t)-\tilde{B}_j(t)][\tilde{B}_i(t)-\tilde{B}_j(t)]^{\mathrm{T}}.\tag{7.36}$$

(Note that (7.34)–(7.36) hold due to the fact that $\mathfrak{G}$ is an undirected graph, and hence, $j\in\mathcal{N}(i)$ if and only if $i\in\mathcal{N}(j)$.) Hence, (7.33) implies that the solution $(E(t),\tilde{\mathbb{A}}(t),\tilde{\mathbb{B}}(t))$ of the parameter error system is Lyapunov stable for all $(E_0,\tilde{\mathbb{A}}_0,\tilde{\mathbb{B}}_0)\in\mathbb{R}^{nN}\times\mathbb{R}^{nN\times n}\times\mathbb{R}^{nN\times m}$ and $t\ge 0$.

Next, note that (7.33) implies

$$V(E(0),\tilde{\mathbb{A}}(0),\tilde{\mathbb{B}}(0)) \geq V(E(t),\tilde{\mathbb{A}}(t),\tilde{\mathbb{B}}(t)) + \lambda_{\min}(R)\int_0^t \sum_{i=1}^N \|e_i(\tau)\|_2^2 \mathrm{d}\tau$$
$$+ \int_0^t \sum_{i=1}^N \sum_{j\in\mathcal{N}(i)} \|e_{ij}(\tau)\|_2^2 \mathrm{d}\tau + \int_0^t \sum_{i=1}^N \sum_{j\in\mathcal{N}(i)} \|\tilde{A}_{ij}(\tau)\|_{\mathrm{F}}^2 \mathrm{d}\tau$$
$$+ \int_0^t \sum_{i=1}^N \sum_{j\in\mathcal{N}(i)} \|\tilde{B}_{ij}(\tau)\|_{\mathrm{F}}^2 \mathrm{d}\tau,$$

and hence, $E(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ or, equivalently, $e_i(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $i = 1,\ldots,N$, with $\hat{A}_i(\cdot) \in \mathcal{L}_\infty$, $\hat{B}_i(\cdot) \in \mathcal{L}_\infty$, $e_{ij}(\cdot) \in \mathcal{L}_2$, $\hat{A}_{ij}(\cdot) \in \mathcal{L}_2$, and $\hat{B}_{ij}(\cdot) \in \mathcal{L}_2$, $i,j = 1,\ldots,N$. Furthermore, since $\hat{A}_i(\cdot)$ and $\hat{B}_i(\cdot)$ are bounded, it follows that $\hat{A}_{ij}(\cdot) \in \mathcal{L}_\infty$ and $\hat{B}_{ij}(\cdot) \in \mathcal{L}_\infty$, $i,j = 1,\ldots,N$. Now, since $x(t)$, $t \geq 0$, and $u(t) \geq 0$, are bounded, it follows from (7.29) that $\dot{e}_i(\cdot) \in \mathcal{L}_\infty$, and hence, by Barbalat's lemma [31, p. 221] (since $e_i(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e}_i(\cdot) \in \mathcal{L}_\infty$) it follows that $\lim_{t\to\infty} \|e_i(t)\|_2 = 0$, $i = 1,\ldots,N$.

Next, since $e_i(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e}_i(\cdot) \in \mathcal{L}_\infty$, it follows that $e_{ij}(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e}_{ij}(\cdot) \in \mathcal{L}_\infty$, and hence, $\lim_{t\to\infty} \|e_{ij}(t)\|_2 = \lim_{t\to\infty} \|\hat{x}_{ij}(t)\|_2 = 0$, $i,j = 1,\ldots,N$. Finally, it follows that $\dot{\hat{A}}_{ij}(\cdot)$ and $\dot{\hat{B}}_{ij}(\cdot)$ are bounded, since $x(t)$, $t \geq 0$, and $u(t)$, $t \geq 0$, are bounded, and thus, by Barbalat's lemma [31, p. 221], $\lim_{t\to\infty} \|\hat{A}_{ij}(t)\|_{\mathrm{F}} = 0$ and $\lim_{t\to\infty} \|\hat{B}_{ij}(t)\|_{\mathrm{F}} = 0$. □

The proposed adaptive consensus distributed observers given in Theorem 7.4.1 guarantee state and parameter estimate consensus as well as convergence of the pairwise difference of the adaptive estimates. This follows as a direct consequence of the $\mathcal{L}_2$ boundnedness of the pairwise disagreement of the parameter estimates.

Next, to examine the dynamic agreement of the parameter and state estimates, we consider the error system (7.29)–(7.31) in a compact form. Specifically, define $\mathbb{J} \triangleq \mathcal{L} \otimes I_n$ so that (7.20)–(7.22) become

$$\dot{E}(t) = \mathbb{A}_{\mathrm{m}} E(t) + \tilde{\mathbb{A}}(t)x(t) + \tilde{\mathbb{B}}(t)u(t) - \mathbb{P}^{-1}\mathbb{J}E(t), \quad E(0) = E_0, \quad t \geq 0, \quad (7.37)$$

$$\dot{\tilde{\mathbb{A}}}(t) = -\Gamma_{\mathrm{a}}\mathbb{P}E(t)x^{\mathrm{T}}(t) - \Gamma_{\mathrm{a}}\mathbb{J}\tilde{\mathbb{A}}(t), \quad \tilde{\mathbb{A}}(0) = \tilde{\mathbb{A}}_0, \tag{7.38}$$

$$\dot{\tilde{\mathbb{B}}}(t) = -\Gamma_{\mathrm{b}}\mathbb{P}E(t)u^{\mathrm{T}}(t) - \Gamma_{\mathrm{b}}\mathbb{J}\tilde{\mathbb{B}}(t), \quad \tilde{\mathbb{B}}(0) = \tilde{\mathbb{B}}_0. \tag{7.39}$$

Equivalently, (7.37)–(7.39) can be rewritten in operator form as

$$\begin{bmatrix} \dot{E}(t) \\ \dot{\tilde{\mathbb{A}}}(t) \\ \dot{\tilde{\mathbb{B}}}(t) \end{bmatrix} = \left(\tilde{\mathfrak{A}}(x(t), u(t)) - \tilde{G}\tilde{J}\right) \begin{bmatrix} E(t) \\ \tilde{\mathbb{A}}(t) \\ \tilde{\mathbb{B}}(t) \end{bmatrix}, \quad \begin{bmatrix} E(0) \\ \tilde{\mathbb{A}}(0) \\ \tilde{\mathbb{B}}(0) \end{bmatrix} = \begin{bmatrix} E_0 \\ \tilde{\mathbb{A}}_0 \\ \tilde{\mathbb{B}}_0 \end{bmatrix}, \quad t \geq 0, \tag{7.40}$$

where

$$\tilde{G} \triangleq \begin{bmatrix} \mathbb{P}^{-1} & 0 & 0 \\ 0 & \Gamma_{\mathrm{a}} & 0 \\ 0 & 0 & \Gamma_{\mathrm{b}} \end{bmatrix}, \quad \tilde{J} \triangleq \begin{bmatrix} \mathbb{J} & 0 & 0 \\ 0 & \mathbb{J} & 0 \\ 0 & 0 & \mathbb{J} \end{bmatrix}.$$

Note that (7.40) has a similar structure to (7.23), differing only in the additional term $\tilde{J}$ which enforces consensus.

To assess the convergence properties of the deviation of (7.14)–(7.16) from the mean, let

$$\delta_{ie}(t) = e_i(t) - \frac{1}{N}\sum_{j=1}^{N} e_j(t) = \frac{1}{N}\sum_{j\neq i}^{N} e_{ij}(t),$$

with analogous expressions for $\delta_{i\mathrm{a}}(t)$ and $\delta_{i\mathrm{b}}(t)$. The convergence of the deviation from the mean of $e_i(t)$, $\tilde{A}_i(t)$, and $\tilde{B}_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, can now be established using the fact that the pairwise disagreement of the state and parameter errors converge to zero, and hence, $\lim_{t\to\infty} \delta_{ie}(t) = 0$, $\lim_{t\to\infty} \delta_{i\mathrm{a}}(t) = 0$, and $\lim_{t\to\infty} \delta_{i\mathrm{b}}(t) = 0$, $i = 1, \ldots, N$. This implies that the individual deviations of the adaptive state and parameter estimates from their mean (static average) converge to zero. This is summarized in the following proposition.

Proposition 7.4.1. Consider the dynamical system (7.1) with A and B unknown. Assume that $\mathfrak{G}$ defines a connected undirected graph of N agents implementing the distributed adaptive consensus observers (7.26)–(7.28), and let the deviations of the state and parameter estimates of the observers from their mean (static average) be given by

$$\delta_{ie}(t) = \hat{x}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{x}_j(t) = e_i(t) - \frac{1}{N}\sum_{j=1}^{N} e_j(t) = \frac{1}{N}\sum_{j\neq i}^{N}\hat{x}_{ij}(t) = \frac{1}{N}\sum_{j\neq i}^{N} e_{ij}(t),$$

$$\delta_{i\mathrm{a}}(t) = \hat{A}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{A}_j(t) = \tilde{A}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\tilde{A}_j(t) = \frac{1}{N}\sum_{j\neq i}^{N}\hat{A}_{ij}(t) = \frac{1}{N}\sum_{j\neq i}^{N}\tilde{A}_{ij}(t),$$

$$\delta_{i\mathrm{b}}(t) = \hat{B}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{B}_j(t) = \tilde{B}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\tilde{B}_j(t) = \frac{1}{N}\sum_{j\neq i}^{N}\hat{B}_{ij}(t) = \frac{1}{N}\sum_{j\neq i}^{N}\tilde{B}_{ij}(t).$$

Then, $\lim_{t\to\infty}\|\delta_{ie}(t)\|_2 = 0$, $\lim_{t\to\infty}\|\delta_{i\mathrm{a}}(t)\|_2 = 0$, and $\lim_{t\to\infty}\|\delta_{i\mathrm{b}}(t)\|_2 = 0$, for $i = 1, \ldots, N$.

Proof. The proof is a direct consequence of Theorem 7.4.1 by noting the pairwise convergence $\lim_{t\to\infty}\hat{x}_{ij}(t) = 0$, $\lim_{t\to\infty}\hat{A}_{ij}(t) = 0$, and $\lim_{t\to\infty}\hat{B}_{ij}(t) = 0$, and the fact that $N < \infty$. □

Alternatively, one can also consider the deviations from the mean of the estimates of the neighboring agents. Specifically, defining

$$\gamma_{ie}(t) \triangleq \frac{1}{\deg(i)}\sum_{j\in\mathcal{N}(i)} e_{ij}(t), \quad \gamma_{i\mathrm{a}}(t) \triangleq \frac{1}{\deg(i)}\sum_{j\in\mathcal{N}(i)}\tilde{A}_{ij}(t),$$
$$\gamma_{i\mathrm{b}}(t) \triangleq \frac{1}{\deg(i)}\sum_{j\in\mathcal{N}(i)}\tilde{B}_{ij}(t), \tag{7.41}$$

we can relate these expressions to the graph Laplacian of $\mathfrak{G}$. In particular, let

$$\gamma_e(t) = \begin{bmatrix}\gamma_{1e}(t)\\ \gamma_{2e}(t)\\ \vdots \\ \gamma_{Ne}(t)\end{bmatrix}, \quad \gamma_{\mathrm{a}}(t) = \begin{bmatrix}\gamma_{1\mathrm{a}}(t)\\ \gamma_{2\mathrm{a}}(t)\\ \vdots \\ \gamma_{N\mathrm{a}}(t)\end{bmatrix}, \quad \gamma_{\mathrm{b}}(t) = \begin{bmatrix}\gamma_{1\mathrm{b}}(t)\\ \gamma_{2\mathrm{b}}(t)\\ \vdots \\ \gamma_{N\mathrm{b}}(t)\end{bmatrix},$$

and note that, since $\sum_{j\in\mathcal{N}(i)} E_{ij}(t)$, $\sum_{j\in\mathcal{N}(i)}\tilde{A}_{ij}(t)$, and $\sum_{j\in\mathcal{N}(i)}\tilde{B}_{ij}(t)$ correspond to the ith block-row of $\mathbb{J}E(t)$, $\mathbb{J}\tilde{\mathbb{A}}(t)$, and $\mathbb{J}\tilde{\mathbb{B}}(t)$, respectively, it follows that $\gamma_e(t) = (\Delta^{-1}\otimes I_n)\,\mathbb{J}E(t)$, $\gamma_{\mathrm{a}}(t) = (\Delta^{-1}\otimes I_n)\,\mathbb{J}\tilde{\mathbb{A}}(t)$, and $\gamma_{\mathrm{b}}(t) = (\Delta^{-1}\otimes I_n)\,\mathbb{J}\tilde{\mathbb{B}}(t)$. Now, it follows from Proposition 7.4.1 that

$$\lim_{t\to\infty}\|\mathbb{J}E(t)\|_{\mathrm{F}} = \lim_{t\to\infty}\|\mathbb{J}\tilde{\mathbb{A}}(t)\|_{\mathrm{F}} = \lim_{t\to\infty}\|\mathbb{J}\tilde{\mathbb{B}}(t)\|_{\mathrm{F}} = 0.$$

It is important to note, however, that $\lim_{t\to\infty} \|\tilde{\mathbb{A}}(t)\|_{\rm F} = 0$ and $\lim_{t\to\infty} \|\tilde{\mathbb{B}}(t)\|_{\rm F} = 0$ cannot be established unless one imposes the additional condition of persistency of excitation. This demonstrates the benefit of information sharing (i.e., graph connectivity), wherein the absence of $\mathbb{J}$ removes the convergence results on consensus unless a persistency of excitation condition is imposed.

7.5. Extensions to Networks with Directed Graph Topologies

In this section, we extend the results of Section 7.4 to adaptive consensus of distributed observers over networks with directed graph topologies.

Theorem 7.5.1. Consider the dynamical system (7.1) with A and B unknown. Assume that $\mathfrak{G}$ defines a weakly connected and balanced directed graph of N agents implementing the distributed adaptive observers given by

$$\dot{\hat{x}}_i(t) = A_{\rm m}\hat{x}_i(t) + (\hat{A}_i(t) - A_{\rm m})x(t) + \hat{B}_i(t)u(t) - P^{-1}\sum_{j\in\mathcal{N}_{\rm in}(i)}(\hat{x}_i(t) - \hat{x}_j(t)),$$
$$\hat{x}_i(0) = \hat{x}_{i0}, \quad t \geq 0, \quad (7.42)$$

$$\dot{\hat{A}}_i(t) = -\Gamma_{{\rm a}i}Pe_i(t)x^{\rm T}(t) - \Gamma_{{\rm a}i}\sum_{j\in\mathcal{N}_{\rm in}(i)}(\hat{A}_i(t) - \hat{A}_j(t)), \quad \hat{A}_i(0) = \hat{A}_{i0}, \quad (7.43)$$

$$\dot{\hat{B}}_i(t) = -\Gamma_{{\rm b}i}Pe_i(t)u^{\rm T}(t) - \Gamma_{{\rm b}i}\sum_{j\in\mathcal{N}_{\rm in}(i)}(\hat{B}_i(t) - \hat{B}_j(t)), \quad \hat{B}_i(0) = \hat{B}_{i0}, \quad (7.44)$$

where $i = 1, \ldots, N$ and P satisfies (7.6). Then, the solution $(E(t), \tilde{\mathbb{A}}(t), \tilde{\mathbb{B}}(t))$ of the parameter error system is Lyapunov stable for all $(E_0, \tilde{\mathbb{A}}_0, \tilde{\mathbb{B}}_0) \in \mathbb{R}^{nN} \times \mathbb{R}^{nN\times n} \times \mathbb{R}^{nN\times m}$ and $t \geq 0$, and $\lim_{t\to\infty} \hat{x}_{ij}(t) = 0$, $\lim_{t\to\infty} \hat{A}_{ij}(t) = 0$, $\lim_{t\to\infty} \hat{B}_{ij}(t) = 0$, $i, j = 1, \ldots, N$, and $\lim_{t\to\infty} e_i(t) = 0$, $i = 1, \ldots, N$.

Proof. Given (7.42)–(7.44) the state and parameter error dynamics satisfy

$$\dot{e}_i(t) = A_{\rm m}e_i(t) + \tilde{A}_i(t)x(t) + \tilde{B}_i(t)u(t) - P^{-1}\sum_{j\in\mathcal{N}_{\rm in}(i)}e_{ij}(t), \quad e_i(0) = e_{i0},$$

$$t \geq 0, \quad (7.45)$$

$$\dot{\tilde{A}}_i(t) = -\Gamma_{\mathrm{a}i} P e_i(t) x^{\mathrm{T}}(t) - \Gamma_{\mathrm{a}i} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \tilde{A}_{ij}(t), \quad \tilde{A}_i(0) = \tilde{A}_{i0}, \quad (7.46)$$

$$\dot{\tilde{B}}_i(t) = -\Gamma_{\mathrm{b}i} P e_i(t) u^{\mathrm{T}}(t) - \Gamma_{\mathrm{b}i} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \tilde{B}_{ij}(t), \quad \tilde{B}_i(0) = \tilde{B}_{i0}, \quad (7.47)$$

for $i = 1, \ldots, N$. Next, consider the distributed Lyapunov function candidates given by (7.24) and note that the derivatives of $V_i(e_i, \tilde{A}_i, \tilde{B}_i)$, $i = 1, \ldots, N$, along the trajectories of (7.45)–(7.47) are given by

$$\begin{aligned}\dot{V}_i(e_i(t), \tilde{A}_i(t), \tilde{B}_i(t)) = & -e_i^{\mathrm{T}}(t) R e_i(t) - 2 e_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} e_{ij}(t) \\ & -2 \operatorname{tr}\left[\tilde{A}_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \tilde{A}_{ij}(t)\right] - 2 \operatorname{tr}\left[\tilde{B}_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \tilde{B}_{ij}(t)\right]. \quad (7.48)\end{aligned}$$

Since $\mathfrak{G}$ is balanced, $\deg_{\mathrm{in}}(i) = \deg_{\mathrm{out}}(i)$, $i = 1, \ldots, N$, and hence, it follows that

$$\begin{aligned}\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \|e_i(t) - e_j(t)\|_2^2 &= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (e_i^{\mathrm{T}}(t) e_i(t) - 2 e_i^{\mathrm{T}}(t) e_j(t) + e_j^{\mathrm{T}}(t) e_j(t)) \\ &= \sum_{i=1}^{N} (\deg_{\mathrm{in}}(i) + \deg_{\mathrm{out}}(i)) e_i^{\mathrm{T}}(t) e_i(t) \\ &\quad -2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} e_i^{\mathrm{T}}(t) e_j(t) \\ &= 2 \sum_{i=1}^{N} \deg_{\mathrm{in}}(i) e_i^{\mathrm{T}}(t) e_i(t) - 2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} e_i^{\mathrm{T}}(t) e_j(t) \\ &= 2 \sum_{i=1}^{N} e_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} e_i(t) - 2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} e_i^{\mathrm{T}}(t) e_j(t) \\ &= 2 \sum_{i=1}^{N} e_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (e_i(t) - e_j(t)). \quad (7.49)\end{aligned}$$

Next, note that since $\mathfrak{G}$ is balanced, the following identities hold

$$2 \operatorname{tr} \sum_{i=1}^{N} \tilde{A}_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (\tilde{A}_i(t) - \tilde{A}_j(t))$$

$$= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \mathrm{tr}[\tilde{A}_i(t) - \tilde{A}_j(t)][\tilde{A}_i(t) - \tilde{A}_j(t)]^{\mathrm{T}}, \tag{7.50}$$

$$2\,\mathrm{tr} \sum_{i=1}^{N} \tilde{B}_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (\tilde{B}_i(t) - \tilde{B}_j(t))$$
$$= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \mathrm{tr}[\tilde{B}_i(t) - \tilde{B}_j(t)][\tilde{B}_i(t) - \tilde{B}_j(t)]^{\mathrm{T}}. \tag{7.51}$$

Now, using (7.48)–(7.51) it follows that

$$\begin{aligned}
&\dot{V}(E(t), \tilde{\mathbb{A}}(t), \tilde{\mathbb{B}}(t)) \\
&\quad = \sum_{i=1}^{N} \dot{V}_i(e_i(t), \tilde{A}_i(t), \tilde{B}_i(t)) \\
&\quad = -\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) R e_i(t) - 2 \sum_{i=1}^{N} e_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (e_i(t) - e_j(t)) \\
&\qquad -2\,\mathrm{tr}\left[\sum_{i=1}^{N} \tilde{A}_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (\tilde{A}_i(t) - \tilde{A}_j(t))\right] \\
&\qquad -2\,\mathrm{tr}\left[\sum_{i=1}^{N} \tilde{B}_i^{\mathrm{T}}(t) \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} (\tilde{B}_i(t) - \tilde{B}_j(t))\right] \\
&\quad = -\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) R e_i(t) - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}(i)} \|e_{ij}(t)\|_2^2 - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \|\tilde{A}_{ij}(t)\|_{\mathrm{F}}^2 \\
&\qquad - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \|\tilde{B}_{ij}(t)\|_{\mathrm{F}}^2 \\
&\quad \leq 0, \quad t \geq 0,
\end{aligned} \tag{7.52}$$

which shows that the solution $(E(t), \tilde{\mathbb{A}}(t), \tilde{\mathbb{B}}(t))$ of the parameter error system is Lyapunov stable for all $(E_0, \tilde{\mathbb{A}}_0, \tilde{\mathbb{B}}_0) \in \mathbb{R}^{nN} \times \mathbb{R}^{nN \times n} \times \mathbb{R}^{nN \times m}$ and $t \geq 0$. The remainder of the proof now follows using identical arguments as in the proof of Theorem 7.4.1 and, hence, is omitted. □

It is important to note that an identical result to Proposition 7.4.1 also holds for the distributed adaptive observers given by (7.42)–(7.44) with a weakly connected and balanced directed graph communication topology. This is immediate from Theorem 7.5.1 using the fact that the pairwise disagreement of the state and parameter errors converge to zero.

7.6. Illustrative Numerical Example

In this section, we present a numerical example to demonstrate the utility and efficacy of the proposed adaptive estimation algorithm using multiagent identifiers with interagent communication. Specifically, we consider an aircraft dynamical system representing the controlled longitudinal motion of a Boeing 747 airplane linearized at an altitude of 40 kft and a velocity of 774 ft/sec given by ([10])

$$\dot{x}(t)=\begin{bmatrix}-0.003 & 0.039 & 0 & -0.332\\ -0.065 & -0.319 & 7.74 & 0\\ 0.02 & -0.101 & -0.429 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}x(t)+\begin{bmatrix}0.01\\ -0.18\\ -1.16\\ 0\end{bmatrix}u(t),\quad x(0)=0,\quad t\geq 0,\quad (7.53)$$

where $x(t)=[x_1(t),x_2(t),x_3(t),x_4(t)]^{\mathrm{T}}$, $t\geq 0$, is the system state vector with $x_1(t)$, $t\geq 0$, representing the x-body-axis component of the velocity of the aircraft center of mass with respect to the reference axes (in ft/sec), $x_2(t)$, $t\geq 0$, representing the z-body-axis component of the velocity of the aircraft center of mass with respect to the reference axes (in ft/sec), $x_3(t)$, $t\geq 0$, representing the y-body-axis component of the angular velocity of the aircraft (pitch rate) with respect to the reference axes (in crad/sec), $x_4(t)$, $t\geq 0$, representing the pitch Euler angle of the aircraft body axes with respect to the reference axes (in crad), and $u(t)$, $t\geq 0$, representing the elevator input (in crad). Figure 7.1 shows the system response to a doublet input.

For our first simulation, we assume that the system matrices A and B characterizing (7.53) are unknown and consider the system (7.53) with four agent identifiers defined by a connected undirected graph topology implementing the distributed adap-

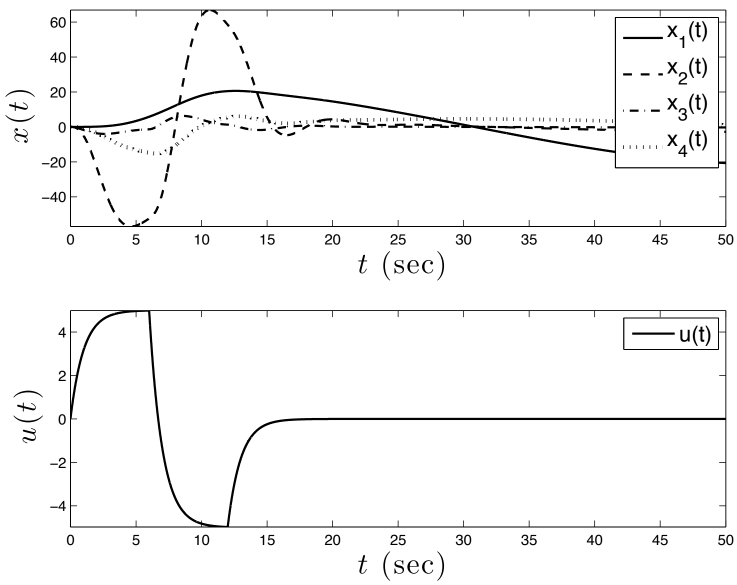

Figure 7.1. System response and doublet input for Boeing 747.

tive observers given by (7.26)–(7.28). Furthermore, we set $A_{\rm m} = -10I_4$, $\Gamma_{\rm a1} = \Gamma_{\rm b1} = I_4$, $\Gamma_{\rm a2} = \Gamma_{\rm b2} = 2I_4$, $\Gamma_{\rm a3} = \Gamma_{\rm b3} = 4I_4$, and $\Gamma_{\rm a4} = \Gamma_{\rm b4} = 8I_4$. Figures 7.2 and 7.3 show the system response for $e_i(t)$, $t \geq 0$, $\hat{x}_{ij}(t)$, $t \geq 0$, $\hat{A}_{ij}(t)$, $t \geq 0$, and $\hat{B}_{ij}(t)$, $t \geq 0$, $i = 1, \ldots, 4$, which, by Theorem 7.4.1, guarantees that $\lim_{t\to\infty} e_i(t) = 0$, $\lim_{t\to\infty} \hat{x}_{ij}(t) = 0$, $\lim_{t\to\infty} \hat{A}_{ij}(t) = 0$, and $\lim_{t\to\infty} \hat{B}_{ij}(t) = 0$, $i, j = 1, \ldots, 4$, without imposing a persistency of excitation condition on the input $u(t)$, $t \geq 0$.

Next, we consider a network of agent identifiers with a strongly connected and balanced directed graph topology shown in Figure 7.4. The parameters used for our simulation are identical to the ones used for the undirected graph topology case. Figures 7.5 and 7.6 show the system response for $e_i(t)$, $t \geq 0$, $\hat{x}_{ij}(t)$, $t \geq 0$, $\hat{A}_{ij}(t)$, $t \geq 0$, and $\hat{B}_{ij}(t)$, $t \geq 0$, $i = 1, \ldots, 4$, which, by Theorem 7.5.1, guarantees that $\lim_{t\to\infty} e_i(t) = 0$, $\lim_{t\to\infty} \hat{x}_{ij}(t) = 0$, $\lim_{t\to\infty} \hat{A}_{ij}(t) = 0$, and $\lim_{t\to\infty} \hat{B}_{ij}(t) = 0$.

Finally, to assess the efficacy of the proposed approach we compare our adaptive

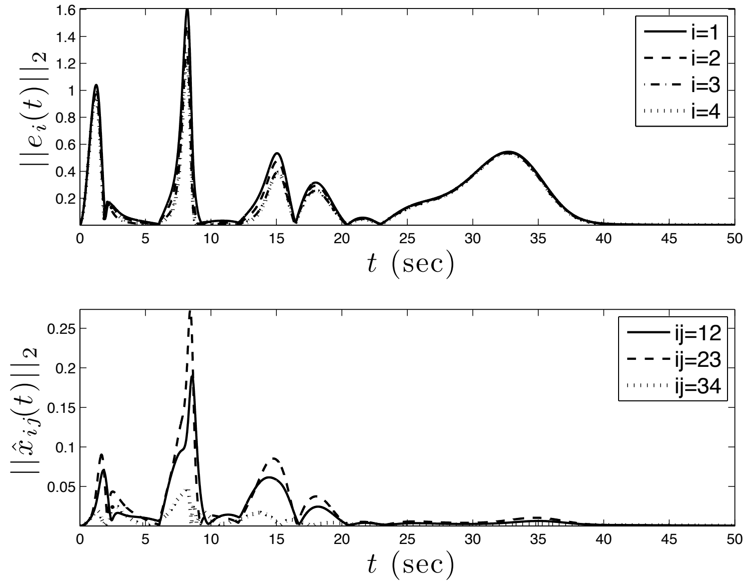

Figure 7.2. State error $\|e_i(t)\|_2$ and $\|\hat{x}_{ij}(t)\|_2$ versus time for the proposed distributed adaptive observers given by (7.26)–(7.28).

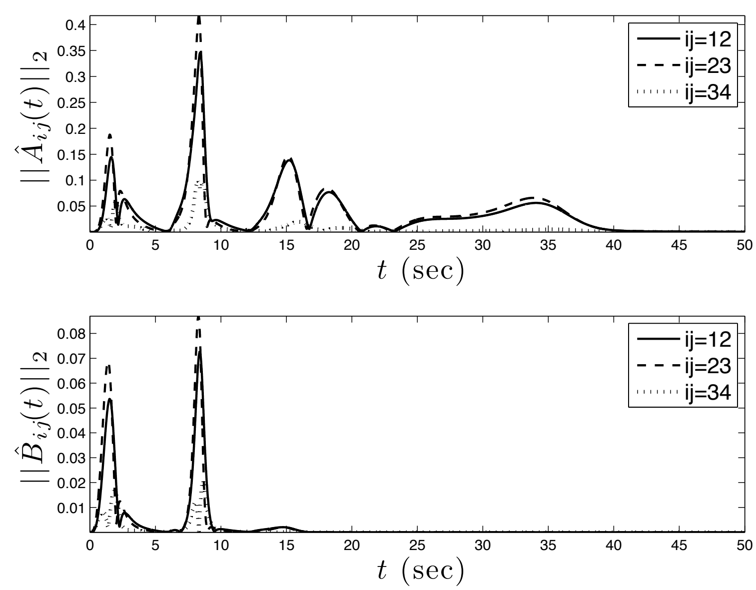

Figure 7.3. Estimate differences $\|\hat{A}_{ij}(t)\|_{\mathrm{F}}$ and $\|\hat{B}_{ij}(t)\|_{\mathrm{F}}$ versus time for the proposed distributed adaptive observers given by (7.26)–(7.28).

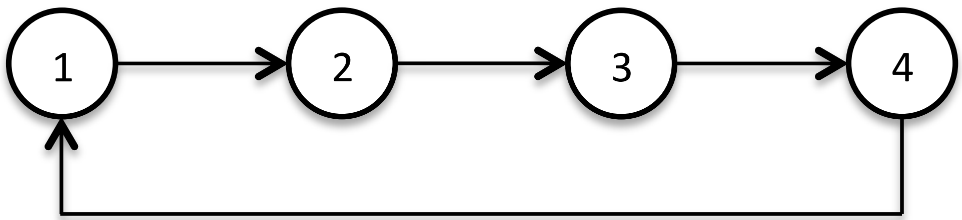

Figure 7.4. Interagent communication graph topology.

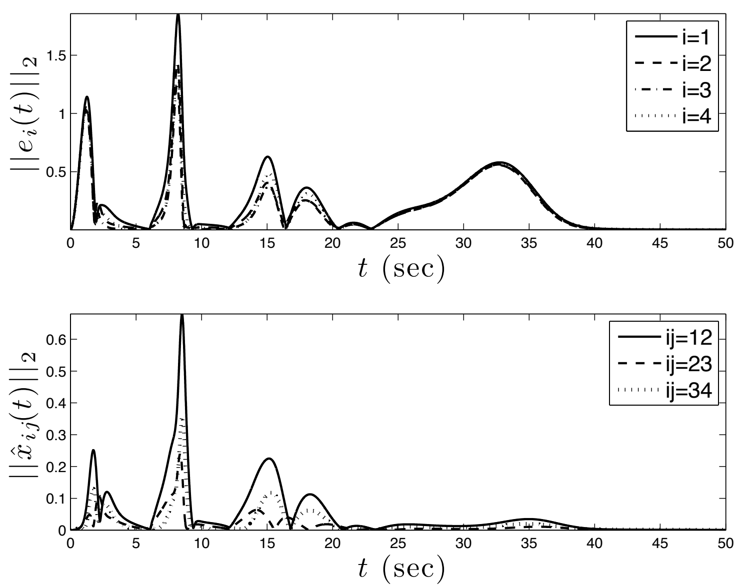

Figure 7.5. State error $\|e_i(t)\|_2$ and $\|\hat{x}_{ij}(t)\|_2$ versus time for the proposed distributed adaptive observers given by (7.42)–(7.44).

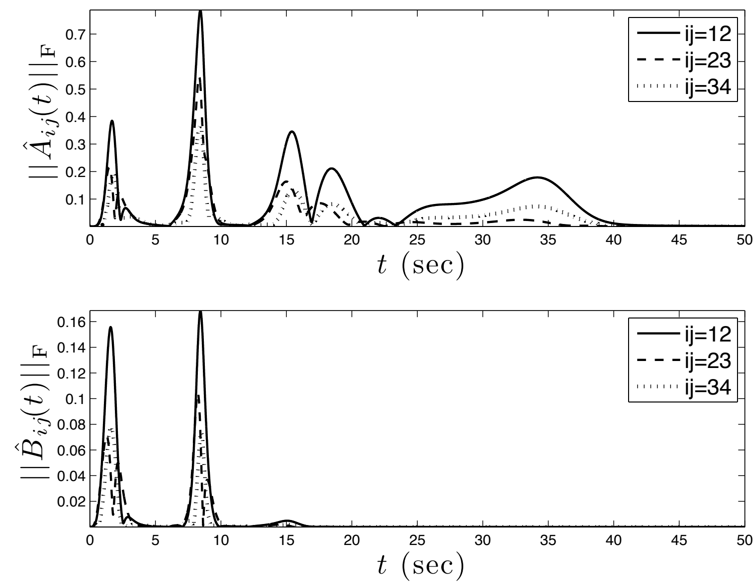

Figure 7.6. Estimate differences $\|\hat{A}_{ij}(t)\|_{\mathrm{F}}$ and $\|\hat{B}_{ij}(t)\|_{\mathrm{F}}$ versus time for the proposed distributed adaptive observers given by (7.42)–(7.44).

estimation multiagent network framework with the standard centralized estimator (7.2)–(7.5). Here, we set $\Gamma_{\mathrm{a}} = \Gamma_{\mathrm{b}} = 1$. Figure 7.7 shows the system response for $e(t)$, $t \geq 0$. It can be seen from this figure that the distributed adaptive estimator (Figures 7.2 and 7.5) significantly outperforms the centralized adaptive estimator.

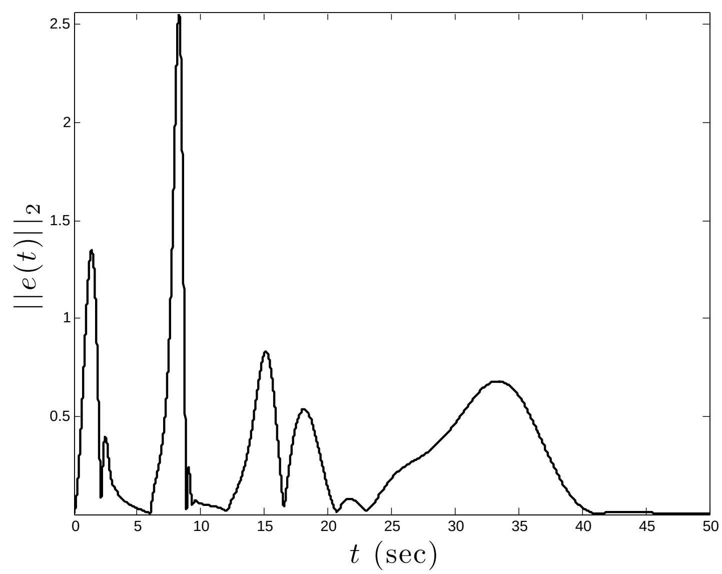

Figure 7.7. State error $\|e(t)\|_2$ versus time for the centralized adaptive observer given by (7.2)–(7.5).

Chapter 8

Conclusion and Ongoing Research

8.1. Conclusion

In this dissertation, we extended the notion of dissipativity theory for continuous dynamical systems with continuously differentiable flows to discontinuous dynamical systems whose solutions are characterized by Filippov set-valued maps. Furthermore, extended Kalman–Yakubovich–Popov conditions in terms of the discontinuous system dynamics for characterizing dissipativity via generalized Clarke gradients of locally Lipschitz continuous storage functions were developed. These results are then used to develop feedback interconnection stability results for discontinuous systems thereby providing a generalization of the small gain and positivity theorems to systems with discontinuous vector fields.

In addition, sufficient conditions for gain, sector, and disk margin guarantees for discontinuous nonlinear systems controlled by nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion were derived. Using these results, connections between dissipativity and optimality of discontinuous nonlinear systems were established. These results provide a generalization of the meaningful inverse optimal nonlinear regulator stability margins as well as the classical linear-quadratic optimal regulator gain and phase margins to discontinuous nonlinear regulators.

Many systems that possess smooth control Lyapunov functions do not necessarily admit a continuous stabilizing feedback controller, even though a stabilizing continuous feedback controller guarantees the existence of a smooth control Lyapunov function. However, the existence of a control Lyapunov function allows for the design of a stabilizing feedback controller that admits Filippov and Krasovskii closed-loop system solutions. In this dissertation, we developed a constructive universal feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives thereby addressing the problem of discontinuous stabilization for dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps and admitting Filippov solutions. In the case where the system vector field is locally Lipschitz continuous and our control Lyapunov function is assumed to be continuously differentiable, our results specialize to the control Lyapunov function of Artstein [3] and our constructive universal controller specializes to Sontag's universal feedback control law [68].

By extending the classical results on dissipativity, optimality and control of systems with continuously differentiable flows to systems with discontinuous vector fields, the results developed in this dissertation provide the tools that can be used to address difficult problems in switched and non-smooth systems such as dynamical networks with switching topologies and spacecraft docking applications.

Furthermore, in this dissertation, we considered a multiagent consensus problem in which agents possess sensors with limited accuracy. In numerous large-scale network system as well as robotics applications, agents can detect the location of the neighboring agents only approximately due to low sensor quality or detrimental environmental conditions. In such settings, it is desirable that the agents reach consensus approximately. We developed consensus control protocols for continuous- and discrete-time network systems that guarantee that the agents reach an almost consensus state and converge to a set centered at the centroid of the agents' initial locations thereby addressing the problem of consensus when agents have limited sensor accuracy. This set is shown to be time-varying, in the sense that only the differences between agent positions are, in the limit, small. In addition, we presented a formulation of the problem using set-valued maps and a set-valued invariance principle. This approach can be used to study different problems in multiagent systems such as area coverage and pursuit evasion under limited sensor accuracy.

Finally, we considered an adaptive estimation problem using a class of multiagent systems serving as adaptive identifiers using an undirected and directed communication graph topology. Specifically, the proposed adaptive architecture includes a modification to the standard adaptive law for distributed adaptive observers which penalizes the pairwise disagreement of the parameter adaptive estimates. This architecture ensures that both the state and parameter estimates reach consensus. We showed that there is no guarantee that the N adaptive identifiers which have the same structure but do not share information will have their estimates converge to the same value without a persistency of excitation condition being imposed. However, the update laws for the parameter identifiers developed in this dissertation by including interagent information exchange guarantee consensus of both the state and parameter estimates, and thus, emulate a persistency of excitation condition.

8.2. Recommendations for Future Research

One of the challenges encountered in numerous engineering applications, e.g., nonsmooth impact systems, systems with shocks, switched and networked systems, is the discontinuous nature of the underlying dynamics. In future research, we propose to use the framework for stability, dissipativity, and optimal control of discontinuous dynamical systems developed in this dissertation to develop control design protocols for dynamical networks with switching topologies involving state-dependent communication links for addressing information link failures and communication dropouts, which can be modeled as discontinuous dynamics. Specifically, we propose to analyze general consensus protocols for multiagent systems of the form given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N,$$
$$u_i(t) = \sum_{j \in \mathcal{N}_{\rm in}(i)} \phi_{ij}(x_i(t), x_j(t)),$$

where $\phi_{ij}(\cdot, \cdot)$, $i, j = 1, \ldots, N$, are Lebesgue measurable and locally essentially bounded; e.g., $\phi_{ij}(x_i, x_j) = \mathrm{sign}(x_j - x_i)$, $i, j = 1, \ldots, N$.

Furthermore, we propose to investigate the application of the newly developed theory to asteroid landing problems since the low-gravity environment exhibited by asteroids poses several challenges due to the discontinuous nature of the impact dynamics during landing. In addition, spacecraft docking applications can also be studied using the newly developed framework due to the discontinuous nature of the underlying dynamics.

Another challenge in robotics applications is dealing with inaccurate sensor data. Specifically, for a group of mobile robots the measurement of the exact location of the other robots relative to a particular robot is often inaccurate due to sensor uncertainty or detrimental environmental conditions. In future research, we propose to use the set-valued framework developed in this dissertation to develop control design protocols for static and dynamic networks with partial directed uncertain interagent communication. Furthermore, we propose to study pursuit evasion problems when agent locations are uncertain using a set-valued analysis framework.

Finally, we propose to investigate border patrolling and area coverage under sensor uncertainties using the techniques developed in this dissertation. To elucidate this problem, let N agents belong to a bounded, convex environment $Q \subset \mathbb{R}^n$ [17]. The position of ith agent is denoted $x_i \in Q$. Furthermore, let $\{V_1, \ldots, V_N\}$ be the Voronoi partition of Q [17], for which the agent positions are the generator points. Specifically, for $i, j = 1, \ldots, N$,

$$V_i \triangleq \{p \in Q : \|p - x_i\|_2 \leq \|p - x_j\|_2, \forall j \neq i\}.$$

Then, for each agent $i \in \{1, \ldots, N\}$, the coverage problem under uncertain sensor measurements consists of calculating its own Voronoi partition in a distributed sense using only local information. This takes into account sensory function which can be thought of as a weighting of importance over Q, and knowing the location of agent j up to an accuracy of a ball of radius r centered at the actual location of agent j.

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